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Elastic Wave Propagation In Inhomogeneous Lossy Media With Special Reference to Powders

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Preface

This report presents a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Connecticut 1980.

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Elastic wave propagation in actual fact is the transport of energy in a medium that acts as a sink for this energy. Engineering and physical applications rely upon an understanding of the wave propagation process to model systems and describe the physical world, and, as such one desires to broaden his knowledge of the actual propagation process in real inhomogeneous media.

In this work, on examines the vector wave equation developed for an anisotropic, inhomogeneous lossy medium with a set of "effective" Lame' constants derived from microscopic considerations of the medium and develops (over)

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a composite loss function, L, to account for the energy loss mechanisms that occur as the wave propagates. This loss function is incorporated into the general equations of motion and series solutions for bulk, and surface waves in the inhomogeneous lossy medium are derived. The displacement fields are examined over the frequency regime in the near and far field for the waves. The effective Lamé parameters are included in the solution, and their contribution, as well as the frequency contribution to the wave field is seen. Higher orders of scattering and diffraction are seen as well as the anharmonicites and nonlinear effects of losses. These effects may be observed in the general wavefield description. Also the effect of losses on the dispersion behavior is observed as a frequency shift of the wave. ⟵

The specific losses for an inhomogeneous granular medium (powder) surrounded by fluid (liquid/gas/or vacuum) are developed and ordered in magnitude over the frequency spectrum. These losses are then incorporated into the general solution and a specific solution for the granular powder is obtained. In addition, further specific examples including source function and loss function incorporation for a nonlinear lossy situation for geophysical and engineering instances are handled by the general method with boundary conditions for finite media. Finally, a comparison of the range of validity of perturbation theory as opposed to the generalized solution in nonlinear series is made and crossover criteria with regard to loss mechanisms operating (frequency considerations) is made. The general method may be applied to acoustics, electromagnetics and seismic wave studies by the appropriate recasting of the governing equations of motion and development of the functional forms for the losses in the medium considered, as well as the generation of the correct constitutive relationships.

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LIST OF SYMBOLS

S = scattering loss

F = frictional loss

V = viscous loss

T = thermodynamic loss

A = intrinsic attenuation in solid

B = intrinsic attenuation in the liquid/gas

U = displacement vector = (u,v,w)

$\bar{\mu},\bar{\lambda}$ = effective Lamé constants (from microscopic theory)

L = generalized loss function

$\bar{v}$ = particle velocity

q,k = complex wave vectors $(k = k_1 + ik_2)(q = q_1 + iq_2)$

ρ = density

t = time

Am,Bq = nonlinear coefficients in generalized wave series

R,m,q = integer counters (subscripted counters)

ω = angular frequency

$i = \sqrt{-1}$

U_o = amplitude of wave

x,y,z = spatial variable

λ = wavelength

D_N = nonlinear attenuation order interaction coefficient (of order N)

LIST OF SYMBOLS

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E = energy

C = speed of sound

μ' = coefficient of friction

$\bar{S}$ = strain field

T' = stress field

P = pressure

Y = yield stress of granules

Ac = contact areas of granules

d = indentation distance of granules

α = angle

R = "radius" (size) of granule

A_t = total surface area of granules

$\bar{V}$ = volume of granule

$\bar{V}_{i,j}$ = volume of medium i or j as indicated

e_{ij} = strain component

n = number of contact points

K = incompressibility modulus

ϕ, θ = angles

U_{pot} = potential energy

M_i = mass of ith granule

f' = general force symbol

η = viscosity

μ = coefficient of friction between granules

LIST OF SYMBOLS

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$$\cdot = \frac{\partial}{\partial t}$$

γ = Gruneisen parameters

τ = relaxation time

a, a^* = creation and destruction operations used in scattering processes

L' = line element of constant $\omega = \omega'$ in k' space

K_1, K_2 = thermal conductivities of media 1 or 2

Q = heat flow

$\overline{T}$ = temperature

S' = entropy

λ' = bulk modulus

Cv = specific heat at constant volume

α' = diffusion constant

Cp = specific heat at constant pressure

β = expansivity

$\overline{r}' = (C_p/C_v)$

$\overline{f}$ = frequency

F = stress dyadic = $F_x\overline{i} + F_y\overline{j} + F_z\overline{k}$

N = phonon population number

$\overline{n}$ = small deviation in equilibrium occupation number $(N = N^o + \overline{n})$

N^o = equilibrium phonon population number

g = acceleration due to gravity

l = mean free path

r = range distance between centers of particles

CHAPTER I

INTRODUCTION

For the most part, theoretical work on elastic wave
propagation has dealt with the development of solutions
that do not adequately (if at all) account for the energy
loss mechanisms that take place. Indeed work on scattering
has been done, from a correlation point of view (Chernov,
1960) and from an intrinsic attenuation point (Klemens,
1955). The nonlinear occurrences and effects of strong
scattering as well as the formulation and identification
and actual development of losses and microscopic mechanisms
has not been handled. Up to now layering to linearize
the wave equation has been dealt with by the linear theory
(Kravtsov, 1968) (Brekhovskikh, 1955, 1960). Mathematical
solutions to the wave equation have been developed in
terms of displacement potentials but have not included
loss mechanisms (Ewing, Jadetsky and Press, 1957). The
classical theory of attenuation has been produced but
it has not included non-linear effects. Losses have not
been used in an idealized model (Auld, 1973) (Brillouin
1946, 1960). Wave propagation in inhomogeneous media
has been approached from a geometrical description of
particle stacking and size rather than from a more

sophisted mathematico-physical standpoint (Brutsaert,
1964; Bit, 1961). Similar approaches in linearization of
the prob and omission of any other losses but scattering
has been lowed (Christian, 1973; Egorov, 1961).

Multe order scattering interactions are not handled
completelSkudrzyk, 1957) while an idealized Green's
function ution with no losses enumerated has been proposed
(Beaudet, 70). Finally, Burger's equation has been
dealt withn a estimated manner in handling lightly dis-
sipative mda (Leibavich and Seebass, 1974). Consequently,
solutions scribing progagation lead to idealized behavior
that can, ia number of instances, depart from observed
phenomena so nuch that the solution's value is depleted.

This problem has been undertaken because one desires
to formulate theories in conformity with reality as much
as possible. A definition of energy transport in the
light of dissipative energy mechanisms occurring in the
propagation process, leads one to the point of developing
a solution that incorporates the loss mechanisms in its
functional form and orders these dissipative processes
over the frequency regions. In this way, the propagation
including dispersion, loss mechanisms, and waveform behavior
is examined and presented in a functional form. One can
then deduce certain physical phenomena that are actually
observed in the propagation process as a result of having
the generalized solution, such as damping of electrical
signals in lossy materials, whereas an idealized solution

omits these processes. Dominant loss mechanisms occurring

over the frequency spectrum are now investigated and one

is provided with insight into the physical mechanisms

occurring during the propagation process. In this treatment,

a granular, inhomogeneous lossy medium is considered,

but the method developed is applicable to a wide spectrum

of propagation problems that occur in acoustics, elasticity,

electromagnetics, and geophysics.

CHAPTER II

Formulation and Development of Generalized Solution

for Equations Governing Elastic Wave Propagation

in an Arbitrary Inhomogeneous Lossy Medium.

As any elastic disturbance propagates through a lossy, inhomogeneous medium of arbitrary composition, there are several sources of energy loss that occur. For an inhomogeneous granular medium composed of solid and fluid components, one encounters the following loss mechanisms.

S - (1) Scattered energy out from the primary wave due to inhomogeneities.

F - (2) Intergranular sliding (frictional loss due to motion induced by passing wave).

V - (3) Viscous energy dissipated into fluid by the motion of granules relative to the fluid.

T - (4) Thermodynamic heat loss, (Energy given up by the wave to the medium as heat as it goes through a pressure cycle, heat conduction)

A - (5) Intrinsic Attenuation in the Solid (rate process).

B - (6) Intrinsic Attenuation in the Liquid (rate process).

As a result of these energy losses, the propagation of a wave will be modified from that of the lossless case. In order then to describe the actual propagation in its entirety, it becomes necessary to examine the effect of the lossy mechanisms that are taking place inside the medium. ⎯

To begin, one must obtain the displacement field in the medium. The approach for doing this is to employ the vector wave equation for inhomogeneous media with a generalized loss term, L, as the starting point. The loss function L, is incorporated into the equation of motion and a series solution is developed in the parameter (kx) where $k = k_1 + ik_2$ is the complex wave vector; from this, one may examine the solution and express it as a modification from the basic exponential solution for a lossless medium.

The equation of motion for an anisotropic inhomogeneous medium as derived in Appendix 4 in its full form with a generalized loss term, L, is the following:

$$\nabla \left[(\bar{\lambda} + 2\bar{\mu}) \nabla \cdot U \right] - \nabla \times \left(\bar{\mu} \nabla \times U \right) + 2 \left[(\nabla \bar{\mu} \cdot \nabla U) - \nabla \bar{\mu} (\nabla \cdot U) \right]$$

$$+ 2 \left[\nabla \bar{\mu} \times (\nabla \times U) \right] = \rho \frac{\partial^2 U}{\partial t^2} + \underline{L(\underline{k}, t)}$$

(2.1)

(1) In general λ and μ are fourth order tensors for an anisotropic medium, however, for an arbitrary medium the wave field will interact with the medium as if a set of "effective" Lamé constants $\bar{\lambda}$, $\bar{\mu}$ were present. Hence we define effective μ anisotropic, λ anisotropic as $\bar{\lambda}$, $\bar{\mu}$ that will subsequently be developed from the microscopic theory to be used in the development.

where $U = (u,v,w)$ is the general displacement vector
and $\bar{\lambda}$ and $\bar{u}$ are the effective Lamé parameters.

This equation balances the net forces and accelerations
of a small volume element of the medium. In the vector
loss term $(L/\bar{v})$, $\bar{v}$ is the particle velocity (a complex
number), while L is the energy lost by the wave per unit
volume per unit time due to the various attenuation processes.
Here L must always be a positive quantity, and $L/\bar{v}$ is
an effective vector force per unit volume, always opposing
the instantaneous motion. The quantity L is additively
composed of contributions from the undivided loss mechanisms.

In order to develop insight into the processes taking
place in the lossy medium, one considers a quasi-one dimen-
sional problem,[2] thus with $\bar{v} = (x,t)$, $\bar{\mu} = \bar{\mu}(x)$, $\bar{\lambda} = \bar{\lambda}(x)$ a plane-wave solution is examined whereby the fundamental
physics lies in the solution to the one-dimensional lossy
problem. Equation (2.1) thus becomes, noting that the displace-
ment and effective elastic parameters are now functions
of one variable, x,

$$\frac{\partial}{\partial x}\left(\bar{\lambda} + 2\bar{\mu}\right)\frac{\partial U}{\partial x} + 2\left[\nabla\bar{\mu}\cdot\frac{\partial U}{\partial x} - \nabla\bar{\mu}\frac{\partial U}{\partial x}\right]$$

$$+ 2\left(\nabla\bar{\mu} \times \left(\frac{\partial V}{\partial z} - \frac{\partial U}{\partial y}\right)\right) = \varrho\frac{\partial^2 U}{\partial t^2} + \left(\frac{L}{\bar{v}}\right)$$

(2.2)

[2] A full three dimensional solution is obtainable
through the use of complex convolution of one dimensional
solutions.

From the above, because of the functional dependences, the second and third terms vanish identically so that the equation reduces to:

$$\frac{\partial}{\partial x}\left(\left(\bar{\lambda}+2\bar{\mu}\right)\frac{\partial U}{\partial x}\right) = \ell\,\frac{\partial^2 U}{\partial t^2} + \angle\left(\frac{\partial t}{\partial U}\right)$$

(2.3)

which may be written in the form,

$$\left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial \bar{\lambda}}{\partial x}\right) + 2\left(\frac{\partial \bar{\mu}}{\partial x}\right)\left(\frac{\partial U}{\partial x}\right) = \ell\,\frac{\partial^2 U}{\partial t^2}$$
$$+ \angle\left(\frac{\partial t}{\partial U}\right) - \left(\frac{\partial^2 U}{\partial x^2}\right)\left(\bar{\lambda}+2\bar{\mu}\right)$$

(2.4)

This partial differential equation is then regrouped as:

$$\left(\ell\,\frac{\partial^2 U}{\partial t^2}\right) = \left(\frac{\partial^2 U}{\partial x^2}\right)\left(\bar{\lambda}+2\bar{\mu}\right) + \left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial \bar{\lambda}}{\partial x}\right) + \left(\frac{\partial \bar{\mu}}{\partial x}\right)\left(\frac{\partial U}{\partial x}\right)$$
$$- \angle\left(\frac{\partial t}{\partial U}\right)$$

(2.5)

Because, in the approximate low loss case, the series expansion of the damped plane wave solution is composed of constants times terms in (kx) (as seen in Appendix I), one surmises that the displacement, U(x,t) in general complex, can be expanded into a general series solution of the functional form:

$$U = \left(U_x\right)\mathcal{e}^{iwt} = U_0\,\mathcal{e}^{iwt}\left[\sum_m A_m\,(kx)^m + i\sum_q B_q\,(kx)^q\right]$$

(2.6)

Where m and q are integer counters and the m are
even and the q are odd, corresponding to the even and
odd terms in the general plane wave solution expansion.
This general form (2.6) is prompted by the "basic function"
$e^{i(k_1 + ik_2)x} e^{i\omega t}$ for weak (and linear) attenuation process,
but is modified to account for the loss of a more general
character.

Substituting the trial form into the lossy equation
one obtains two systems of equations in terms of coef-
ficients of powers of (kx), through separating real and
imaginary terms, namely setting reals equal to reals and
imaginaries equal to imaginaries.

Now, in the weak, linear attenuation case the coefficients
A_m and B_q are constants (as seen in Appendix 1) corresponding
to the expansion of the classical exponential form. Now,
depending on the nature of L these coefficients themselves
become functions of k and x. The quantity k is analogous
to the wave number in the ordinary case. Every wave of
frequency ω is assumed to have associated with it a parameter
k. After factoring out the oscillatory time dependent
term $e^{i\omega t}$, to allow for isolation of the dependence due
to damping one develops a power series solution in (kx).
From the general form, cited in (2.6), one notes that
$k = k_1 + ik_2$ because of the loss mechanism, so that, sub-
stituting this form into (2.6) there arises,

$$(U_x) = U_0 \left[\sum_m A_m \left((k_1 + ik_2)^m x^m \right) + (i) \sum_q B_q \left((k_1 + ik_2)^q x^q \right) \right] \tag{2.7}$$

so that, upon regrouping terms, one has,

$$\left(U_x\right) = \left[\left(\sum_m A_m \left(\sum_{N=0}^{m} k_1^{M-N} \frac{(ik_2)^N}{N!} ((M) \cdot \cdot \cdot \cdot (M-N+1)) \right) x^M \right. \right.$$

$$\left. \left. + (i) \sum_q B_q \left(\sum_{R=0}^{q} (k_1^{q-R})(q \cdot \cdot \cdot \cdot q-R+1) \frac{(ik_2)^R}{R!} x^q \right) \right] U_0 \right.$$

(2.8)

For convenience in notation, one writes,

$$\binom{q}{R} = \left(\frac{(q) \cdot \cdot \cdot \cdot \cdot \cdot (q-R+1)}{R!} \right)$$

$$\binom{M}{N} = \left(\frac{(M) \cdot \cdot \cdot \cdot \cdot \cdot (M-N+1)}{N!} \right)$$

(2.9)

It is possible to simplify to the following, casting the equation into the binomial expansion format:

$$\left(U_x\right) = U_0 \left[\sum_m A_m \left(\sum_{N=0}^{m} (k_1^{M-N})(k_2)^N (i^N)(x^M)\binom{M}{N} \right) \right.$$

$$\left. + (i) \sum_q B_q \left(\sum_{R=0}^{q} (k_1)^{q-R}(k_2)^R (i^R)(x^q)\binom{q}{R} \right) \right]$$

(2.10)

One segregates U_x into real and imaginary parts. From this formulation for U_x as a series one substitutes the form into the generalized equation of motion with losses and obtains the nonlinear coefficients A_m and B_q solved as recursion coefficients. Segregating U_x into real and imaginary parts, one has

$$\left(U_x\right) = U_a\left[\sum_m \sum_{\substack{N'=2N=0\\ \text{even }N'}}^{m} A_m\,(k_1)^{M-N'}(k_2)^{N'}\left(i^{N'}\right)\left(x^m\right)\binom{m}{N} \quad (2.11)\right.$$

$$+ \sum_m \sum_{\substack{N''=2N-1=1\\ \text{odd }N'}}^{m} A_m\,(k_1{}^{M-N''})(k_2)^{N''}\left(i^{N''}\right)\left(x^m\right)\binom{m}{N}$$

$$+ (i)\sum_q \sum_{\substack{R'=2R=0\\ \text{even }R'}}^{q} B_q(k_1)^{q-R'}(k_2)^{R'}\left(i^{R'}\right)\left(x^q\right)\binom{q}{R}$$

$$\left. + (i)\sum_q \sum_{\substack{R''=2R-1=1\\ \text{odd }R''}}^{q} B_q(k_1)^{q-R''}(k_2)^{R''}\left(i^{R''}\right)\left(x^q\right)\binom{q}{R}\right]$$

Now the general form is convergent, since, for the
weak attenuation case the classical form is convergent,
and the surmised series is an extension of this form.
It is itself a modified exponential expanded into series
form. Since the wave decays in the weak attenuation case,
and one considers even stronger attenuation mechanisms
occuring, the wave most certainly decays and its solution
form must be a series that converges at least as fast
as the desired solution, $(e^{ik_1x}\,e^{i\omega t}\,e^{-k_2x})\,U_o$.

Recalling the general form for the equation of motion
in the lossy, inhomogeneous medium, one obtains from Eq.
(2.4) in quasi-one dimensional form:

$$\left(\bar{\lambda}' + 2\bar{\mu}'\right)\left(\frac{\partial U}{\partial x}\right) + \left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 U}{\partial x^2}\right) = \rho\left(\frac{\partial^2 U}{\partial t^2}\right) + L\left(\frac{\partial t}{\partial U}\right)$$

In which

$$\bar{\lambda} = \bar{\lambda}_{Ave} + \Delta\bar{\lambda}(x)\ ; \qquad \bar{\mu} = \bar{\mu}_{Ave} + \Delta\bar{\mu}(x)$$

$$\Delta\bar{\lambda}(x) \longrightarrow 0\ ; \qquad \Delta\bar{\mu}(x) \longrightarrow 0 \quad \text{large } x$$

$$\text{implies} \quad \left(\bar{\lambda}' + 2\bar{\mu}'\right) \longrightarrow \left(\bar{\lambda}'_{Ave} + 2\bar{\mu}'_{Ave}\right) \longrightarrow 0$$

$$\bar{\lambda}' = \left(\partial\bar{\lambda}/\partial x\right) \qquad \bar{\mu}' = \left(\partial\bar{\mu}/\partial x\right)$$

Eq. (2.4) then becomes[3]:

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 U_x}{\partial x^2}\right) + \rho\, U_x \omega^2 = \frac{L}{e^{+i\omega t}}\left(\frac{\partial t}{\partial U}\right)$$

(2.12)

One sees that (2.12) is the resulting quasi-one dimensional partial differential equation incorporating the effect of losses. Upon substituting Eq. (2.7) into (2.12) and separating real and imaginary terms, one arrives at the following set of equations:

Equating the real parts, one obtains:

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left[(m)(m-1)\sum_{m}\sum_{N'=2N=0}^{m} A_m (k_1^{m-N'})(k_2)^{N'}(i^{N'})(x^{m-2})\binom{m}{N}\right.$$

$$\left. + q(q-1)\left(\sum_{q}\sum_{R''=2R-1=1}^{q} B_q (k_1)^{q-R''}(k_2)^{R''}(i^{R''+1})(x^{q-2})\binom{q}{R}\right)\right]$$

(2.13)

$$+\left(\omega^2\rho\right)\left[\sum_{m}\sum_{N'=2N=0} A_m(k_1)^{m-N'}(k_2)^{N'}(i^{N'})(x^{m})\binom{m}{N}\right]$$

$$+\left(\omega^2\rho\right)\left[\sum_{q}\sum_{R''=2R-1=1}^{q} B_q(k_1)^{q-R''}(k_2)^{R''}(i^{R''+1})(x^{q})\binom{q}{R}\right]$$

$$= Re\left(\frac{e^{-i\omega t}\,L}{U\,U_0}\right) = Re\left(\frac{L\,e^{i k x}}{-i\omega\,U_0}\right) = \left(\frac{L \sin(kx)}{-\omega\,U_0^2}\right)$$

[3] In most physical situations $\bar{\lambda}$ and $\bar{\mu}$ vary in an irregular manner and oftentimes are discontinuous at the interface between dissimilar materials in an inhomogeneous medium. Hence this approach is applicable to the majority of cases. For the extreme, where $\bar{\lambda}$ and $\bar{\mu}$ are gentle functions of position, an additional term in the series solution appears that causes some secondary effects on scattering.

While equating the imaginary parts, one obtains:

$$(\bar{\lambda} + 2\bar{\mu}) \left[m(m-1) \sum_m \sum_{N''=1}^{m} A_m (k_1^{M-N''})(k_2^{N''})(i^{N''})(x^{M-2})\binom{M}{N}\right.$$

$$\left. + g(g-1) \sum_g \sum_{R'=2R=0}^{g} B_g (k_1)^{g-R'}(k_2)^{R'}(i^{R'+1})(x^{g-2})\binom{g}{R}\right]$$

$$+ (\omega^2 \rho) \left[\sum_m \sum_{N''=1}^{m} A_m (k_1^{M-N''})(k_2)^{N''}(i^{N''})(x^M)\binom{M}{N}\right]$$

$$+ (\omega^2 \rho) \left[\sum_g \sum_{R'=2R=0}^{g} B_g (k_1)^{g-R'}(k_2)^{R'}(i^{R'+1})(x^g)\binom{g}{R}\right]$$

$$= Im\left(\frac{e^{-i\omega t} L}{J U_0}\right) = Im\left(\frac{L e^{ikx}}{i\omega U_o^2}\right) = \left(\frac{(\cos kx) L}{-\omega U_o^2}\right)$$

$$(2.14)$$

Next it is necessary to collect similar terms in the resulting series in powers of x and shift indices in the various summations to obtain recursion relationships among the respective coefficients for like powers of x. It is also necessary to adjust subscripts of coefficients appropriately.

Once again, upon equating real parts there is obtained:

$$(\bar{\lambda} + 2\bar{\mu}) \left[(m)(m-1) \left(\sum_{m=2} \sum_{N'=0}^{m} A_m (k_1^{M-N'})(k_2)^{N'}(i^{N'})(x^{M-2})\binom{M}{N}\right.\right.$$

$$\left. + (g)(g-1) \left(\sum_{g=3} \sum_{R''=2R-1=1}^{g} B_g (k_1)^{g-R''}(k_2)^{R''}(i^{R''+1})(x^{g-2})\binom{g}{R}\right]$$

$$+ (\omega^2 \rho) \left[\sum_{m=2} \sum_{N'=2N=0}^{m} A_{m-2}(k_1)^{M-N'-2}(k_2)^{N'}\binom{M-2}{N}(i^{N'})(x^{M-2})\right]$$

$$+ (\omega^2 \rho) \left[\sum_{g=3} \sum_{R''=2R-1=1}^{g} (B_{g-2})(k_1^{g-R''-2})(k_2)^{R''}(x^{g-2})(i^{R''+1})\binom{g-2}{R}\right]$$

$$= \left[\frac{L (x^N) \sin kx}{-\omega U_o^2}\right]$$

$$(2.15)$$

Similarly, for the imaginary components, one obtains:

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left[(m)(m-1)\sum_{m}\sum_{N''=2N-1=1}^{m} A_m\left(k_1^{m-N''}\right)(k_2)^{N''}\binom{m}{N}(i^{N''})(x^{m-2})\right]$$

$$+ \left(\bar{\lambda} + 2\bar{\mu}\right)\left[q(q-1)\sum_{q=3}\sum_{R'=2R=0}^{q} B_q(k_1)^{q-R'}(k_2)^{R'}\binom{q}{R}(i^{R'+1})(x^{q-2})\right]$$

$$+ w^2\varrho\left[\sum_{m=2}\sum_{N''=2N-1=1}^{m}(A_{m-2})\left(k_1^{m-2-N''}\right)(k_2)^{N''}\binom{m-2}{N}(x^{m-2})(i^{N''})\right]$$

$$+ w^2\varrho\left[\sum_{q=3}\sum_{R'=2R=0}^{q} B_{q-2}(k_1)^{q-2-R'}\binom{q-2}{R}(k_2)^{R'}(i^{R'+1})(x^{q-2})\right]$$

$$= \left[\frac{(\cos kx)\,L\,(x^N)}{-w\,U_0^2}\right] \tag{2.16}$$

So that equating imaginary components, one has symbolically and operationally,

$$R(k)\,A_m\left(x^{m-2}\right) + P(k)\,B_q\left(x^{q-2}\right)$$

$$+ (w^2\varrho)(A_{m-2})\left(x^{m-2}\right)M(k) + (w^2\varrho)\,E(k)(B_{q-2})\left(x^{q-2}\right)$$

$$= L\left(x^{m-2}\right)\left(\frac{\cos kx}{-w\,U_0^2}\right) \tag{2.17}$$

And, equating real components, one obtains in a similar form:

$$d(k)(A_m)\left(x^{m-2}\right) + f(k)\,B_q\left(x^{q-2}\right) + w^2\varrho\,j(k)(A_{m-2})\left(x^{m-2}\right)$$

$$+ (w^2\varrho)(B_{q-2})\left(x^{q-2}\right)S(k) = \left(\frac{L\left(x^{m-2}\right)\sin kx}{-w\,U_0^2}\right) \tag{2.18}$$

where R(k), P(k), M(k), E(k), d(k), f(k), j(k), S(k),
are the respective polynomial coefficients of the terms

and are identified in Eq. (2.15) and (2.16) from corres-
ponding terms in (2.17) and (2.18). The loss term L is
written this way to show that like powers of x are equated
on both sides.

Now, q = m + 1, since the basic series alternates,
so that one has now, a resulting separated set of recursion
relationships between the even and odd coefficients A_m
and B_q. Upon equating similar powers of x on each side,
one has the desired recursion relationships. In these
recursion relationships one obtains two conditions, one
relates $L(x^{q-2})$ to the Bq terms and the other relates
$L(x^{m-2})$ to the A_m terms. For the general A_m recursion
relationship for the equation for the imaginaries one
has:

$$A_m \left(x^{m-2} \right) R(k) + \omega^2 \rho \left(A_{m-2} \right) \left(x^{m-2} \right) M(k)$$

$$= \left[\frac{\cos(k_x) \angle (x^{m-2})}{-\omega U_o^2} \right]$$

$$(2.19)$$

and also, for the B_q recursion relationship for the imagi-
naries there arise:

$$P(k) B_q \left(x^{q-2} \right) + \omega^2 \rho \, E(k) B_{q-2} \left(x^{q-2} \right)$$

$$= \left[\frac{\cos(k_x) \angle (x^{m-1})}{-\omega U_o^2} \right]$$

$$= \left[\frac{\cos(k_x) \angle (x^{q-2})}{-\omega U_o^2} \right] \qquad (2.20)$$

In addition, for the reals, by the same arguments as were
applied for the imaginaries, one obtains,

$$d(k) A_m (x^{m-2}) + w^2 \rho j(k) A_{m-2} (x^{m-2}) = \left(\frac{\sin kx}{-w \, U_o^2} \right) L(x^{m-2})$$

(2.21)

and,

$$f(k) B_q (x^{q-2}) + w^2 \rho (B_{q-2})(x^{q-2}) S(k)$$
$$= \left(\frac{\sin kx}{-w \, U_o^2} \right) \left(L(x^{q-2}) \right)$$

(2.22)

Now, one constructs the total recursion relationship
for real and imaginary through combinations of like powers
of x for each of the A_m even and B_q odd coefficients.
These equations become the following since the coefficient
of each separate power of x must vanish,

$$\left(A_m (m)(m-1) \sum_{N'=2N=0} (k_1)^{m-N'} (k_2)^{N'} (i^N)(\bar{\lambda} + 2\bar{u}) \binom{m}{N} \right)$$
$$+ (w^2 \rho) \left[\sum_{N'=2N=0}^{m} (k_1)^{m-N'-2} (k_2)^{N'} \binom{q-2}{N} (i^N) \right] A_{m-2}$$
$$= \left[\frac{\cos kx}{-w \, U_o^2} L(x^{m-2}) + \frac{i \sin (kx) L(x^{m-2})}{-w \, U_o^2} \right]$$
$$= - \left(\frac{e^{ikx}}{w \, U_o^2} \right) \left(L(x^{m-2}) \right)$$

(2.23)

$$B_q (q)(q-1) \left(\sum_{R''=2R-1=1}^{q} (k_1)^{q-R''} (k_2)^{R''} \binom{q}{R} (i^{R''+1}) (\bar{\lambda} + 2\bar{u}) \right)$$
$$+ (w^2 \rho) \, B_{q-2} \left(\sum_{R''=2R-1=1}^{q} \binom{q-2}{R} (i^{R''+1}) (k_1^{q-R''-2})(k_2)^{R''} \right)$$
$$= - \left(\frac{e^{ikx}}{w \, U_o^2} \right) L(x^{q-2})$$

(2.24)

And starting with the lowest order coefficients, one obtains the beginning of the recursion relationships:

$$m = 0 : \quad A_0(0) = 0 \qquad A_0 \text{ Arbitrary} \tag{2.25}$$

$$q = 1 : \quad B_1(0) = 0 \qquad B_1 \text{ Arbitrary} \tag{2.26}$$

Hence, A_0 B_1 are chosen as 1 since they are arbitrary.

For the general recursion relationship one has, for $m \geq 2$ for the A_m:

$$(A_m)(m)(m-1) \left[\sum_{N'=0} (k_1)^{M-N'}(k_2)^{N'} \binom{m}{N} (i^{N'})(\bar{\lambda} + 2\bar{\mu}) \right]$$

$$+ (\omega^2 \rho) \left[\sum_{N'=0}^{m} (k_1)^{M-N'-2}(k_2)^{N'} \binom{m-2}{N}(i^{N'}) \right] A_{m-2}$$

$$= \left[-\frac{e^{ikx}}{\omega U_0^2} \ L(x^{M-2}) \right] \tag{2.27}$$

Also, noting $q = m + 1$ one obtains for $q > 2$, for the general Bq

$$B_q(q)(q-1) \left(\sum_{R''=1}^{q} (k_1)^{q-R''}(k_2)^{R''}(i^{R''+1})(\bar{\lambda} + 2\bar{\mu}) \binom{q}{R} \right)$$

$$+ (\omega^2 \rho)(B_{q-2}) \left[\sum_{R''=1}^{q} (i^{R''+1})(k_1^{q-R''-2})(k_2)^{R''} \binom{q-2}{R} \right]$$

$$= \left(\frac{-e^{ikx}}{\omega U_0^2} \right) L(x^{q-2}) \tag{2.28}$$

Solving for the recursion relations for the respective coefficients, from equation (2.27) yields,

$$A_m = \left[\frac{\left(-\frac{e^{ik_x x}}{\omega U_o^2}\right)\left(L\left(x^{m-2}\right)\right) - \omega^2 \rho \, A_{m-2}\left[\sum_{N'=0}^{m-N'-2}(k_1)^{m-N'-2}(k_2)^{N'}\binom{m-2}{N'}(i^{-N'})\right]}{(m)(m-1)\sum_{N'=0}^{m-N'}(k_1)^{m-N'}(k_2)^{N'}\binom{m}{N'}(i^{-N'})(\bar{\lambda}+2\bar{u})} \right] \quad (2.29)$$

and also, from Eq. (2.28) there is obtained:

$$B_q = \left[\frac{\left(-\frac{e^{ik_x x}}{\omega U_o^2}\right)L\left(x^{q-2}\right) - \omega^2 \rho\left[\sum_{R''=1}^{m}(i^{-R''+1})(k_1^{m-R''-1})(k_2)^{R''}\right]B_{q-2}}{q(q-1)\sum_{R''=1}^{m}(k_1)^{q-R''}\binom{q}{R}(k_2)^{R''}(\bar{\lambda}+2\bar{u})(i^{-R''+1})} \right] \quad (2.30)$$

Next, it is necessary to express the general A_m in terms of the base coefficient A_o. This is obtained by determining the product of corresponding linked members in the even term A-series.

$$A_m = \left[\frac{\left(-\frac{e^{ikx}}{\omega v_0^2}\right) L(x^{n-2}) - \omega^2 \varrho \left[\sum_{N=0}^{m} \overset{18}{} (k_1)^{n-N'-2} (k_2)^{N'} \binom{m-2}{N} (i^{N'}) \right] (x)}{\cdots\cdots (x) \left[L(x^{M\cdot N}) \frac{e^{ikr}}{\omega v_0^2} - \omega^2 \varrho K(k) \right]} \right.$$

$$\left. \frac{}{(m)(m-1) \sum_{N'=0}^{N} (k_1)^{n-N'} (k_2)^{N'} (i^{N'}) (\bar{\lambda} + 2\bar{u}) \binom{m}{N}} \right]$$

(2.31)

The same procedure is applied to the B_q for the odd term series:

$$B_q = \left[\frac{\left(-\frac{e^{ikx}}{\omega v_c^2}\right) L(x^{q-2}) - \omega^2 \varrho \left[\sum_{R''=1}^{m} \binom{q-2}{R} (i^{R''+1})(k_1)^{m-2L_1} R \right]}{(x)\cdots\cdots (x) \left[\frac{-e^{ikr}}{\omega} L(x^{q-q}) \cdots \right]} \right.$$

$$\left. \frac{}{q(q-1) \sum_{R''=1}^{q} (k_1)^{q-R''}(k_2)^{R''} (i^{R''+1}) \binom{q}{R}(\bar{\lambda}+2\bar{u})} \right]$$

(2.32)

In the feedback product of descending coefficients A_m the higher order product terms are very small and decrease by $(\bar{\lambda}+2\bar{\mu})^N$ so to first order it is beneficial to simplify slightly and include the dominant terms for each coefficient.

Hence, one has for the A_m even coefficients:

$$A_m \simeq \left[\frac{\left(\frac{-e^{ikx}}{\omega v_0^2}\right) L(x^{M-2})}{(m)(m-1) \sum_{N'=0}^{m} (k_1)^{M-N'} \binom{m}{N} (k_2)^{N'} (i^{N'})(\bar{\lambda}+2\bar{u})} \right]$$

(2.33)

and also for the odd term coefficients, B_q,

$$B_q \simeq \left[\frac{\left(\dfrac{-e^{ikx}}{\omega \bar{U}_0^2}\right) \angle (x\,q-2)}{(q)(q-1)\displaystyle\sum_{R''=1}^{q}(k_1)^{q-R''}(i^{R''+1})\binom{q}{R}(k_2)^{R''}(\bar{x}+2\pi)} \right]$$

$$(2.34)$$

with q = m + 1

These are the nonlinear coefficients A_m, B_q that describe the modification to the basic exponential plane wave solution.

Finally, recasting into the original formulation, one obtains the complete solution for the nonlinear lossy problem. The coefficients A_m and B_q are now nonlinear functions of the wave number k

$$(U) = \left(U_0\, e^{i\omega t}\right)\left(\sum_{\substack{m=0 \\ even}} A_m (kx)^m + (i)\sum_{\substack{q=1 \\ odd}} B_q (kx)^q\right)$$

$$= (e^{i\omega t})\, U_0\left[\, 1 + i(k_1 + ik_2)x\right]$$

$$+ \left(U_0\, e^{i\omega t}\right)\left(\sum_{\substack{m=2 \\ even}} (kx)^m \left[\frac{\left(-e^{ikx}/\omega U_0^2\right)\angle(x\,m-2)}{(m)(m-1)(\bar{x}+2\pi)\left(\sum_{N=0} k_1^{m-N} i^N (k_2)^m \binom{m}{N}\right)}\right]\right)$$

$$+ (i)(e^{i\omega t})(U_0)\left(\sum_{\substack{q=3 \\ odd}} (kx)^q \left[\frac{\left(-e^{ikx}/\omega U_0^2\right)(\angle(x\,q-2))}{(q)(q-1)(\bar{x}+2\pi)\sum_{R''=1}(k_1)^{q-R''}\binom{q}{R}k_2^{R''}}\right]\right)$$

$$(2.35)$$

These nonlinear coefficients A_m and B_q indicate the effect of higher order interaction processes (higher order attenuations), and in the general representation may be redesignated as attenuation order interaction coefficients $D_N(k,t)$. The attenuated terms of Nth order, which in the absence of losses would be the terms in a sinusoidal expansion are now modified by the appropriate coefficient of the loss function L. They describe the general nonlinear behavior that arises in the wave field due to the presence of losses.

The solution converges because it is of the form $e^{ik_1x} e^{-k_2D_Nx}$ that converges. It also goes to 0 as $x \to \infty$ due to the damping term $e^{-k_2D_Nx}$ where the D_N are the general nonlinear coefficients for attenuation order interaction terms in the general representation (See Appendix 1). When the D_N are constants one has the classical linear theory when they are functions of L one has the nonlinear theory for higher order nonlinear processes occurring.

Hence, the general field is of the form:

$$U_x = U_0 e^{i\omega t} \left(e^{-ik_1 x D_N} \right) \left(e^{-k_2 x D_N(k,t,L)} \right)$$

(2.36)

while at higher k, one encounters some waveform lengthening and distortion in the near field.

In general, the nonlinear equation (2.12) will generate harmonics. In particular, if one utilizes first order iteration, harmonics are generated with frequencies depending upon the power of U in the loss function L. In second order iteration, one obtains higher order combinations of these multiples. Hence, the total solution then becomes, the following accounting for harmonic generation, for the composite nonlinear series form:

$$U_D = \sum_{p=0}^{\infty} U_p e^{i\omega_p t}$$

(2.37)

in the discrete case

and

$$U_c = \int_\omega d\omega_p U_p e^{i\omega_p t}$$

(2.38)

Here, each Up represents a general nonlinear series solution for a fixed ω, for the lossy problem.

One notes some physical aspects of the solution:

(1) Dispersion will take place in higher orders as U contains complicated powers of $k = k_1 + ik_2$ so that k_2 will also appear in the real part of the effective k as indicated in the expansion.

(2) Attenuation takes place in higher orders as seen in the infinite series of nonlinear interaction terms incorporating multiple scattering, diffraction effects and general loss processes. These are seen as the powers of the complex k in the expansion, particularly with regard to k_2.

(3)   Incorporated in the loss components from L are
interactions from surface and body waves in the scat-
tering mechanism showing that the propagation and
the loss depend also on the geometrical description
of the medium. This occurs because of the integration
limits on the scattering integrals for S, the scattering
loss term in L.

(4)   The effect of the inhomogeneity of the medium
is to create attenuation by scatttering of the wave.
This can combine with other losses to higher order
in the loss function L. The scattering may generally
be formulated sepa' :ely (as herein by perturbation
theory) and inserted as a term in L.

The physical interpretation of the successive terms
can be seen as:

(1)   The first two terms represent the contribution
to the attenuation mechanism (composite energy loss
process), of the "weak, linear interactions" that
one normally experiences in the classical theory,
whereby energy is extracted at a fixed rate.

(2)   The successive higher order attenuations (modifi-
cations to the classical linear terms) arise due
to the nonlinear mechanical variation of the properties
of the medium and the loss mechanisms occurring in
L, the energy loss function. These arise as has
been we have seen from nonlinear rates of energy

extraction, random multiple scattering processes, nonlinear interaction rates, strong attenuation processes in L that distort the waveform nonlinearly, and general random behavior of the medium.

Now, in the classical theory one assumes that the energy is extracted at a constant rate defined by α and then formulates the functional form

$$\left(\frac{dE}{E}\right) = -2\alpha \, dx \Longrightarrow E(x) = E_0 \, e^{-2\alpha x}$$

(2.39)

when in reality, as the losses become significant, nonlinear effects take place and the fixed, constant energy extraction rate process does not adequately describe the physical situation.

In the weak attenuation case the attenuations can be functions of frequency and the coefficients A_m and B_q are constants (all the same function of frequency). In the more general case however, each coefficient A_m and B_q can be a different function of frequency leading to a nonexponential, or modified exponential decay, and in addition these coefficients can be explicitly amplitude dependent.

Depending on the general nature of L, the energy loss function that describes how the energy is lost from the wave into the medium, one obtains more or less complicated

expressions for the D_N, the attenuation order interaction
coefficients. Thus, for a medium that is strongly lossy,
the D_N are complex in functional form; for a medium with
small weak losses, the classical linear theory with D_N
as constants is sufficient. The nonlinear effects introduce
distortion of the wave profile, (equivalent to the production
of higher harmonics) and because of nonlinear effects
the waveform profile steepens until dissipative effects
become important. These dissipative effects lead to broad-
ening of the wave profile since the higher harmonics are
attenuated more rapidly. These effects finally balance
its nonlinear steepening, whereby the rate of energy conver-
sion from the lower to the higher harmonics is counterbalanced
by the increased dissipation of the higher frequency components;
in this situation the wave form remains constant.

(a) <u>Body Wave Formalism and Displacement Field</u>
<u>with Near and Far Field Behavior</u>

It is now of interest to examine the near and far
field behavior of the wave field, in the light of the
following spatial segregation: λ small$(\lambda < R)$; λ large$(\lambda > R)$;
λ intermediate$(\lambda\ R)$; x large$(x > 10\lambda)$; x small$(x < 10\lambda)$; R
granule size.

(1) <u>(Case I)</u> - (λ large, x small), (Near Field)

For k small, x small, the terms in (kx) are rapidly
converging, and the loss function is weak for small k.

Hence, the first few terms of the series are sufficient to describe the wavefield:

$$U_x \sim \left(1 - k_2 x + D_2(k)(k_2 x)^2\right) e^{i\omega t}\left(e^{i k_1 x}\right) U_0$$

(2.40)

(2) <u>(Case II)</u> - (λ, x Intermediate), -general solution given as equation (2.36) is applicable.

(3) <u>(Case III)</u> - (λ small x small) - (Near Field) -Here for k large and x small, the effects of small x are over-ridden by large k. In this case the terms converge quite rapidly. A few terms are sufficient to describe the field behavior adequately.

$$\left(U_x\right) \sim U_0 \left(e^{i\omega t}\right) \left(e^{i k_1 x D_N}\right)\left(1 - (k_2 x) + D_N(k_2 x)^2\right)$$

(2.41)

(4) <u>(Case IV)</u> - (λ small, x large), (Far Field) - The complete form is needed as the terms in (kx) are significant. The losses are strong. One has in the adopted representation:

$$U_x \sim U_0 \, e^{i\omega t}\left(e^{i k_1 x D_N}\right)\left(e^{-k_2 x D_N}\right)$$

(2.42)

One notes here in the far field, that the wavelength becomes lengthened due to the presence of the nonlinear terms D_N in the $e^{i k_1 x D_N}$ oscillation factor.

(5) <u>(Case V)</u> - (λ large, x large), (Far Field) - This
is the low frequency condition. In the far field averaging
effects of the waveform have taken place, and the waveform
has reached a stable configuration. The solution approaches
the classical solution as nonlinear effects have been
overridden.

$$\left(U_x \right) \sim U_0 e^{-k_2 x} \left(e^{ik_1 x} \right) \left(e^{i\omega t} \right)$$

$$(2.43)$$

As a result of the foregoing considerations one constructs
the following table:

FIELD BEHAVIOR OF SOLUTIONS

TABLE I

(Body Waves with Losses)

(1) λ large, x small

(Near Field)

$$U_x \sim \left(1 - k_2 x + \theta_2(k)(k_2 x)^2 \right) e^{i\omega \tau} e^{ikx}(U_0) \quad (2.44)$$

(2) Intermediate λ, x - General Field Solution

As indicated in equation (2.35)

(3) λ small, x small

(Near Field)

$$\left(U_x \right) \sim U_0 e^{i\omega t} \left(e^{ik_1 x \theta_N} \right) \left(1 - (k_2 x) + \theta_2 (k_2 x)^2 \right) \quad (2.45)$$

(4) λ small, x large

(Far Field)

$$\left(U_x \right) \sim U_0 \left(e^{-k_2 x \theta_N} \right) \left(e^{ik_1 x} \right) \left(e^{i\omega t} \right) \quad (2.46)$$

(5) λ large, x large

(Far Field)

$$U_x \sim U_o \left(e^{-k_2 x}\right)\left(e^{i k_1 x}\right)\left(e^{i\omega t}\right) \qquad (2.47)$$

(b) <u>Surface Wave Formalism and Displacement Field</u>

<u>With Near and Far Field Behavior</u>

The foregoing method may be utilized in other cases. The following has been considered

$$\bar{\mu} = \bar{\mu}(x) \; ; \; \bar{\lambda} = \bar{\lambda}(x) \; ; \; U = U(x,t)$$

At this point one may examine surface waves, whereby for the surface wave case, one has, the following

$$\bar{\mu} = \bar{\mu}(y) \; ; \; \bar{\lambda} = \bar{\lambda}(y) \qquad U = U(x,t) f(y) \qquad (2.48)$$

For this situation the equation of motion becomes

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) \left(\bar{\lambda} + 2\bar{\mu}\right) \left(\frac{\partial U}{\partial x}\right)$$

$$= \rho \frac{\partial^2 U}{\partial t^2} + \frac{L}{V} \qquad (2.49)$$

because

$$\nabla \times \left(\bar{\mu}(y) \frac{\partial f}{\partial y}\right) = 0 = 2\left[\nabla\bar{\mu} \cdot \frac{\partial U}{\partial x} - \nabla\bar{\mu} \frac{\partial U}{\partial x}\right]$$

$$\nabla \times \left(\bar{\mu} \frac{\partial U}{\partial z}\right) = 0 = 2\left(\nabla\bar{\mu}(y) \times \frac{\partial f}{\partial y}\right)$$

considering this equation (2.49) becomes:

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 \bar{U}}{\partial x^2}\right) + \left(\frac{\partial \bar{\lambda}}{\partial y} + 2\frac{\partial \bar{\mu}}{\partial y}\right)\left(\frac{\partial \bar{U}}{\partial x}\right)$$
$$+ \left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 \bar{U}}{\partial x \partial y}\right) = \rho\frac{\partial^2 \bar{U}}{\partial t^2} + \left(\frac{L}{\bar{U}}\right)$$

(2.50)

Physically, the waves decay in depth, so

$$f(y) = A_1 e^{-c_1 y}$$

Hence, equation (2.50) becomes,

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 \bar{U}}{\partial x^2}\right) + \left(\frac{\partial \bar{\lambda}}{\partial y} + 2\frac{\partial \bar{\mu}}{\partial y}\right)\left(\frac{\partial \bar{U}}{\partial x}\right)$$
$$+ \left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial \bar{U}}{\partial x}\right)\left(-c_1 A_1 e^{-c_1 y}\right)$$
$$= \rho\left(\frac{\partial^2 \bar{U}}{\partial t^2}\right) + \left(\frac{L}{\bar{U}}\right)$$

(2.51)

regrouping, yields,

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 \bar{U}}{\partial x^2}\right) + \left(\frac{\partial \bar{\lambda}}{\partial y} + 2\frac{\partial \bar{\mu}}{\partial y}\right)\left(\frac{\partial \bar{U}}{\partial x}\right)$$
$$- \left(c_1 A_1 e^{-c_1 y}\right)\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial \bar{U}}{\partial x}\right) = \rho\frac{\partial^2 \bar{U}}{\partial t^2} + \left(\frac{L}{\bar{U}}\right)$$

(2.52)

Again, noting that $\bar{\lambda} = \bar{\lambda}$ ave $+ \Delta \bar{\lambda}(x)$ and $\bar{\mu} = \bar{\mu}$ ave $+ \Delta\bar{\mu}(x)$ and that in the average $\Delta \bar{\lambda}(x) \rightarrow 0$, $(\Delta\bar{\mu}(x))$ $\rightarrow 0$ for an arbitrary medium, so that $(\bar{\lambda}'$ ave $+ 2 \bar{\mu}'$ ave$)$ $\rightarrow 0$, one gets the surface wave equation of motion with losses for the quasi-one dimensional case.[4]

[4]Again, as in the body wave case, a complete three dimensional solution is obtainable through complex convolution of one dimensional solutions.

$$\left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 U}{\partial x^2}\right) - \left(c_1 A_1 e^{-c_1 y}\right)\left(\frac{\partial U}{\partial x}\right)\left(\bar{\lambda} + 2\bar{\mu}\right)$$

$$= \rho \frac{\partial^2 U}{\partial t^2} + \frac{L}{\bar{U}}$$

(2.53)

In order to determine the modified behavior with depth one has to first order, the displacement field in the form

$$U = f(y) U_x = f(y) e^{i k x D_N} e^{-i \omega t}$$

(2.54)

for orthogonal axes x and y and separability.

Placing this in the reduced wave equation (omitting the secondary nonlinear lossy terms), one gets

$$- \omega^2 f(y) e^{i(k x D_N - \omega t)} = c^2 \frac{\partial^2 f}{\partial y^2} e^{i(k x D_N - \omega t)}$$

$$- k^2 D_N^2 f(y) e^{i(k x D_N - \omega t)}$$

(2.55)

or,

$$\left(\frac{\partial^2 f}{\partial y^2}\right) = \left(k^2 D_N^2 - \frac{\omega^2}{c^2}\right) f$$

(2.56)

which arises upon dividing through by the term $(c^2 e^{(i)(k x D_N - \omega t)})$.
Now, if $(k^2 D_N^2 - \frac{\omega^2}{c^2}) < 0$ this equation yields an oscillatory solution that is not damped. This is not possible, as the nonlinear effects alone will cause the wave to decay. Therefore, it is necessary to demand that $(k^2 D_N^2 - \frac{\omega^2}{c^2})$ > 0.

The solution for f is, from the auxiliary equation for Eq. (2.56),

$$f(y) = \overline{C_0}\, e^{\left(\sqrt{k^2 D_N{}^2 - \omega^2/c^2}\right) y}$$

(2.57)

Now, the solution with negative square root would cause the function to increase. This is unacceptable on physical grounds, since y is measured in a negative direction. The square root determine the rapidity of the damping and shows how the nonlinear processes in the losses contribute to the damping.

By a direct transformation, following the development of the previous section, and taking dominant A_m and B_q and regrouping, one has for the field with $\bar{\mu} = \bar{\mu}(y)$, $\bar{\lambda} = \bar{\lambda}(y)$, the form:

$$U_{xy} = f(y)\, U_0 \left(e^{i\omega t}\right)\left(\sum_m (A_m')(kx)^m + (i)\sum_q (B_q')(kx)^q\right)$$

$$= U_0\left(e^{i\omega t}\right)\left(1 + i(k_1 + ik_2)x\right)\left(A_1 e^{-C_1 y}\right)$$

$$+ U_0\left(e^{i\omega t}\right)\left(A_1 e^{-C_1 y}\right)\left[\sum_{\substack{m=2\\even}} \frac{(kx)^m\left(-e^{ikx}/\omega U_0^2\right)L(x^{m-2})}{(\bar{\lambda}+2\bar{\mu})(m)(m-1)\sum_{N'=0}(k_1)^{m-N'}\frac{(k_2)^{N'}}{N'!}\binom{m}{N}}\right]$$

$$+ (i)\left(e^{i\omega t}\right)(U_0)\left[\sum_{\substack{q=3\\odd}} \frac{(kx)^q\left(-e^{ikx}/\omega U_0^2\right)L(x^{q-2})}{(q)(q-1)\sum_{R''=1}(k_1)^{q-R''}(\bar{\lambda}+2\bar{\mu})(k_2)^{R''}\binom{q}{R}}\right]$$

(2.58)

Where it is seen now that the additional y dependence modification of the nonlinear waveform appears. Again,

U(x) can be described as composed of linearly attenuated terms plus higher order terms attenuated to Nth order, while in addition, this horizontal dependence is modified by the nonlinear depth dependence of equation (2.57).

The following table is then constructed for surface waves in the same manner as Table I was formed for body waves with the scale parameters defined in the same way as they were prior.

Field Behavior of Solutions

Table II

(Surface Waves with Losses)
(1) λ small, x small $$U \sim U_0 e^{i\omega t} (A_1 e^{-c_1 y})(e^{ik_1 x D_N})(1 - k_2 x + D_2 (k_2 x)^2) \qquad (2.59)$$
(2) Intermediate   General Solution as Indicated
(3) λ large, x large (Far Field) $$U \sim U_0 e^{-k_1 x}(e^{i(k_1 x)})(e^{i\omega t})(A_1 e^{-c_1 y}) \qquad (2.60)$$
(4) λ large, x small (Near Field) $$U_x \sim U_0 e^{i\omega t}(e^{ik_1 x})(A_1 e^{-c_1 y})(1 - k_2 x + D_2(k)(k_2 x)^2) \qquad (2.61)$$
(5) λ small, x large (Far Field) $$U \sim U_0 e^{i\omega t}(e^{ik_1 x D_N})(e^{-k_2 x D_N})(A_1 e^{-c_1 y}) \qquad (2.62)$$

One notes that the depth dependence of $\bar{\lambda}(y)$ and $\bar{\mu}(y)$ appears in the form of the nonlinear coefficients D_N that govern the damping character of the wavefield.

(c) Effect of Energy Loss on the Dispersion Relationship

It is interesting to determine the effect of losses on the dispersion. One begins with the strain field equation:

$$S = \eta(U)$$

(2.63)

with the general displacement field for losses as:

$$U \sim U_o \, e^{-i\omega t} \, e^{ik_1 x} \, e^{-k_2 x \, D_N}$$

(2.64)

The equation of motion is written as:

$$\left(\bar{\lambda}' + 2\bar{\mu}'\right)\left(\frac{\partial U}{\partial x}\right) + \left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial^2 U}{\partial x^2}\right)$$
$$= \rho\left(\frac{\partial^2 U}{\partial t^2}\right) + \left(\frac{\ell}{\bar{U}}\right)$$

(2.65)

and the constitutive stress-strain equations, in the medium are expressed as:

$$T_i = c_{ij}\,\overline{S_j}$$

(2.66)

for T_i' the stress and $\bar{S}_j$ the strain in indicial notation.

The strain field is then, from Eqs. (2.63) and (2.64),

$$S = \tfrac{1}{2}\left(\frac{\partial U}{\partial x}\right) = i k_1 e^{ik_1 x}\frac{e^{i\omega t}}{2} e^{-k_2 x}D_N$$

$$+ e^{ik_1 x}\frac{e^{i\omega t}}{2} e^{-k_2 x}D_N\left(-k_2 D_N \frac{\partial D_N}{\partial x}\right)$$

which becomes,

$$S = \frac{1}{2}\left(i k_1 - k_2 D_N \frac{\partial D_N}{\partial x}\right)\left[e^{ik_1 x}e^{i\omega t}e^{-k_2 x}D_N\right]$$

$$(2.67)$$

For the general case of loss, the converted equation of motion written in terms of stress and strain becomes from (2.63), (2.65) and (2.66):

$$\left(\bar{\lambda}' + 2\bar{\mu}'\right)\left(2 S\right) + \left(\bar{\lambda} + 2\bar{\mu}\right)\left(\frac{\partial T}{\partial x}\right)$$

$$= \rho\left(\frac{\partial^2 U}{\partial t^2}\right) + \left(\frac{L}{J}\right)$$

$$(2.68)$$

so that the gradient of the stress field is represented by:

$$\left(\frac{\partial T}{\partial x}\right) = c' \frac{\partial^2 U}{\partial x^2} = \left[\left(i k_1 - k_2 D_N \frac{\partial D_N}{\partial x}\right)\right] \cdot$$

$$\cdot\left[i k_1\left(e^{ik_1 x}e^{i\omega t}e^{-k_2 x}D_N\right) + \left(e^{ik_1 x}e^{-k_2 x}D_N e^{i\omega t}\right)\left(-k_2 D_N \frac{\partial D_N}{\partial x}\right)\right]$$

$$+\left[e^{ik_1 x}e^{i\omega t}e^{-k_2 x}D_N\right]\left[\left(-k_2\left(\frac{\partial D_N}{\partial x}\right)^2 - k_2\left(D_N\right)\frac{\partial^2 D_N}{\partial x^2}\right)\right]$$

$$(2.69)$$

and also, noting the following equivalences:

$$\varrho \frac{\partial^2 U}{\partial t^2} = -\varrho \omega^2 e^{ik_1 x} e^{i\omega t} e^{-k_2 x} D_N$$

$$\frac{L}{\bar{V}} = \frac{L}{-i\omega U}$$

Upon substituting (2.67) and (2.69) into the equation of motion for the inhomogeneous lossy medium written in stress and strain field form, one has, the following after cancelling the factor $\left(e^{ik_1 x} e^{i\omega t} e^{-k_2 x D_N} \right)$

$$\left(\bar{\lambda}' + 2\bar{\mu}' \right) \left[ik_1 - k_2 D_N \frac{\partial D_N}{\partial x} \right] + \left(\bar{\lambda} + 2\bar{\mu} \right) \left[ik_1 - k_2 D_N \left(\frac{\partial D_N}{\partial x} \right)^2 \right]$$

$$- \left[k_2 \left(\frac{\partial D_N}{\partial x} \right)^2 + \left(k_2 D_N \left(\frac{\partial^2 D_N}{\partial x^2} \right) \right) \right] \left(\bar{\lambda} + 2\bar{\mu} \right)$$

$$= -\omega^2 - \left(\frac{L}{i\omega U^2} \right)$$

(2.70)

Examining the dominant terms over the frequency regime, with magnitudes considered, and regrouping, one sees,

$$\left(\bar{\lambda}' + 2\bar{\mu}' \right) \left(ik_1 - k_2 D_N \frac{\partial D_N}{\partial x} \right) + \left(\bar{\lambda} + 2\bar{\mu} \right) \left[k_2^2 D_N^2 \left(\frac{\partial D_N}{\partial x} \right)^2 - k_2 \left(\frac{\partial D_N}{\partial x} \right) \right]$$

$$+ \left(\bar{\lambda} + 2\bar{\mu} \right) \left[-\left(k_1^2 + 2i k_1 k_2 D_N \left(\frac{\partial D_N}{\partial x} \right) + \left(k_2 D_N \frac{\partial^2 D_N}{\partial x^2} - k_2 \left(\frac{\partial D_N}{\partial x} \right)^2 \right) \right] \right]$$

$$= -\omega^2 - \left(\frac{L}{i\omega U^2} \right)$$

(2.71)

Hence, one has approximately, with the dominant terms,
noting $(k_1 > k_2)$:

$$-\left(\bar{\lambda}' + 2\bar{\mu}'\right)\left(i k_1\right) + \left(\bar{\lambda} + 2\bar{\mu}\right)\left(k_1\right)^2 = \varrho\, \omega^2 + \frac{L}{i \omega \upsilon^2}$$

Noting,

$$\left(k_1\right)\left(\bar{\lambda}' + 2\bar{\mu}'\right) \ll \left[\left(k_1\right)(k)(\bar{\lambda} + 2\bar{\mu})\right]$$

one has,

$$\varrho\, \omega^3 + \left[-\left(\bar{\lambda} + 2\bar{\mu}\right) k_1^2\right]\omega + \frac{L}{i \upsilon^2} = 0$$

letting,

$$\omega = \omega_0 + \Delta\omega$$

with ω_0, the normal solution, defined as,

$$\varrho\, \omega_0^2 = \left(\bar{\lambda} + 2\bar{\mu}\right) k_1^2$$

Substituting the perturbation representation into
the cubic equation above, one has to second order in $(\Delta\omega)$,
the following equation that arises from dropping higher
order terms:

$$3\varrho\, \omega_0\left(\Delta\omega^2\right) + 2\varrho\, \omega_0^2\left(\Delta\omega\right) = \left(\frac{i L}{\upsilon^2}\right)$$

$$(2.72)$$

Noting that in general $k_2 < k_1$, one obtains the following
equation after having divided (2.72) through by $\left(2\varrho\, \omega_0^2\right)$

$$\left(\tfrac{3}{2}\right)\left(\frac{\Delta\omega^2}{\omega_0}\right) + \Delta\omega = \left(\frac{i L}{2 \upsilon^2 \varrho\, \omega_0^2}\right)$$

$$(2.73)$$

and the roots of (2.73) become:

$$\Delta W_{1,2} = \left(\frac{-2\rho w_0^2 \pm \sqrt{4\rho^2 w_0^4 + \frac{4iL}{v^2}(3\rho w_0)}}{6\rho w_0} \right)$$

$$\Delta W_{1,2} = -\frac{w_0}{3}\left(1 \pm \sqrt{1 + \frac{3iL}{v^2\rho}\left(\frac{w}{w^2}\right)} \right)$$

$$Re(\Delta w) = -\left(\frac{1}{3}\right) w_0 \left(1 \pm \sqrt{1 + 3i\left(\frac{\Delta_0 w}{w}\right)} \right)$$

$$\left(1 + 3i\left(\frac{\Delta_0 w}{w}\right) \right)^{1/2} = 1 + \frac{3}{2}i\left(\frac{\Delta_0 w}{w}\right) + \frac{3}{8}\left(\frac{\Delta_0 w}{w}\right)^2 + \dots$$

$$\simeq 1 + \left(\frac{3}{8}\right)\left(\frac{\Delta_0 w}{w}\right)^2 + \left(\frac{3i}{2}\right)\left(\frac{\Delta_0 w}{w}\right)$$

so that, upon taking the negative root one has,

$$Re(\Delta w) = -\left|\frac{w_0}{3}\right|\left(1 - 1 - \left(\frac{3}{8}\right)\left(\frac{\Delta_0 w}{w}\right)^2 \right) = \left(\frac{w_0}{8}\right)\left(\frac{\Delta_0 w}{w}\right)^2$$

where,

$$Re(\Delta w) = \frac{w_0 \Delta_0 w^2}{8 w^2} \tag{2.74}$$

with,

$$(\Delta_0 w) = \frac{\angle(k)}{v^2 \rho w^2} = \frac{L}{E} \implies \frac{1}{\tau} = \Delta_0 w$$

$$\tag{2.75}$$

Identifying the individual frequency components, one has,

$$W_{total} = \omega_0 + (i \Delta_0 \omega) + Re(\Delta \omega)$$

(2.76)

With

ω_0 = original frequency with no losses

$(i \Delta_0 \omega)$ = loss, frequency becomes imaginary

$Re(\Delta \omega)$ = additional dispersion due to loss, dispersion behavior because of loss

Now, $Re(\Delta \omega)$ and $i(\Delta_0 \omega)$ are frequency shift terms arising because of the loss mechanisms operating. The other root of (2.73) is extraneous, being brought about by the mathematical operations being performed.

One has an equation as:

$$G(U, \omega, k, L) = 0$$

(2.77)

in the dispersion relationship. Since L is an intrinsic function of the amplitude U_0 of the wave train, one actually has

$$F(U_0, k, \omega) = G(U, \omega, k, L) = 0$$

(2.78)

Thus in this case of inhomogeneous lossy medium propagation one has a nonlinear dispersion relationship for the waves and the waveform will be distorted by nonlinear effects taking place at progressive times in the medium as the wave propagates. One sees that, mathematically, losses due to dissipation are manifested by the dispersion relationship yielding complex (or imaginary) values of ω for real values of k.

CHAPTER III

Elastic Wave Propagation in Powders with Particles

Surrounded by Fluids or Vacuum

(a) Development of the Effective Lamé Constants of the

Powder and Environment System

It is now necessary to describe the material properties

of a powder system in terms of the microscopic development

of the effective elastic contants $\bar{\lambda}$ and $\bar{\mu}$ in order to

obtain the general wavefield solution in conjunction

with specific loss mechanisms for the powder and the appropriate

effective elastic constants. One examines the material

properties of a powder system surrounded by vacuum, gas,

or liquid as an environment. In order to determine the

wave velocity in such a powder (inhomogeneous granular

medium) it is necessary to begin by examining the effective

Lamé constants $\bar{\lambda}$, $\bar{\mu}$ obtained from a physical model of

the powder. A general displacement wavefield may also

be obtained for a given system by substitution of the

specific losses and effective elastic constants into the

general solution obtained earlier.

For the incompressibility modulus, K, Y the yield stress,

$\bar{k}'$ the spring constant between particles, n the number

of contact points, P the hydrostatic pressure (equilibrium

pressure), and r the distance between centers of particle
and μ_s the shear modulus of the granules, one has the
ensuing analysis. For the static case, one considers
the hydrostatic pressure acting on a granule of area A_t
having contact area A_c with yield stress Y, ie.,

$$\left(P \, A_t \right) = \left(Y \, A_c \right)$$

(3.1)

and in close approximation to the contact areas for flattened
(rounded) particular one has,

$$A_c = \pi R^2 \sin^2 \alpha \quad \sim \quad \pi R^2 \alpha$$

(3.2)

for R the particle radius and α the angle of intersection.
One uses the small angle approximation for small angles
α. The indentation distance d of the particle due to
pressure P is:

$$d = R \left(\tfrac{1}{2} \alpha^2 \right) \Longrightarrow A_c = 2\pi d R$$

(3.3)

The total area becomes,

$$A_t = 4\pi R^2 + \epsilon R^2$$

(3.4)

for flattened particles where (ϵR^2) represents contact
area. In the linear-volumetric dimension relationship
for the dilatation one has,

$$\left(\frac{\delta \overline{V}}{\overline{V}}\right) = \left(\frac{3\delta d}{R}\right)$$

(3.5)

for the volume $\overline{V}$.

Taking the differential of (3.1) yields:

$$d P (A_t) + P(d A_t) = (Y d A_c)$$

(3.6)

In area-linear relationship one has,

$$\left(\frac{d A_t}{A_t}\right) = \frac{2 d \overline{V}}{\overline{V}} = \left(\frac{2\delta d}{R}\right)$$

(3.7)

Taking the differential of (3.2) yields,

$$d A_c = 2\pi R (\delta d)$$

Dividing by A_t yields

$$\left(\frac{d A_c}{A_t}\right) = 2\pi R \left(\frac{\delta d}{A_t}\right)$$

(3.8)

Dividing the equation (3.6) by (PA_t) yields

$$\left(\frac{dP}{P}\right) + \left(\frac{d A_t}{A_t}\right) = \frac{Y}{P}\left(\frac{d A_c}{A_t}\right)$$

(3.9)

Upon substitution of (dA_c/A_t) from the above and (dA_t/A_t) from above one has

$$\left(\frac{dP}{P}\right) = \left(\frac{Y}{P}\right)\left(\frac{2\pi R^2}{A_t}\frac{\delta d}{\cdot R}\right) - \left(\frac{2}{3}\right)\frac{d\overline{V}}{V}$$

(3.10)

Finally, dividing through by $(d\overline{V}/\overline{V})$ and multiplying (3.11) by P there results, for particles and vacuum:

$$\left(\frac{dP}{d\overline{V}}\right)\overline{V} = K = \left(\frac{Y}{2}\right) = \left(\overline{\lambda} + \frac{2}{3}\overline{\mu}\right)$$

At this point it is necessary to develop the effective $\overline{\mu}$ and $\overline{\lambda}$ for the powder with the various environments. One utilizes the equation

$$E_{total} = E_{shear} + E_{compressional}$$

(3.13)

for the total energy in terms of its shear and compressional components. For an effective spring constant of k between particle bonds, with each particle having n contact points, and noting that $(d\overline{V}/\overline{V})$ is the dilatation then the strain energy transferred becomes on a per particle basis,

$$E_{compressional} = \frac{1}{2}\overline{k}'n\,(\delta r)^2 = \frac{1}{2}\overline{k}'r^2 n\left(\frac{\delta\overline{V}}{3\overline{V}}\right)^2$$

(3.14)

where (δr) is measured in the radial extension and the linear-volumetric relationship is

$$\left(\frac{\delta r}{r}\right) = \left(\frac{\delta \overline{V}}{3\overline{V}}\right)$$

(3.15)

On a volumetric basis, one has,

$$E_{volumetric} = \frac{1}{2} K \left(\frac{\delta \overline{V}}{\overline{V}}\right)^2$$

(3.16)

For the shear term, one notes that the extension along a horizontal axis becomes

$$(\delta r) = (r \cos\theta)\left(\frac{\partial U_y}{\partial x} \cos\theta\right) = (r \cos^2\theta)\left(\frac{\partial U_y}{\partial x}\right)$$

(3.17)

with,

$$E_{total} = E_{shear} + E_{compressional}$$

Combining one has,

$$E_{total} = \frac{1}{2} \overline{k}' n r^2 (\cos^2\theta)(e_{yx})^2 + A_c^{\frac{1}{2}} r^2 (e_{yx})^2_{solid}$$

so that,

$$E_{total} = \frac{1}{2} \overline{u} (e_{yx})^2 \overline{V}_{particle}$$

(3.18)

Since the displacement $\Delta\mu$~re for e the strain on a per bond basis one has with the shear energy

$$E_{shear} = \mu\,(Ac)^{3/2} \left(\frac{\Delta U}{Ac^{1/2}}\right)^2$$

$$= A_c^{1/2}\, r^2\, (e_{yx})^2\, \mu$$

(3.19)

Considering the stored volumetric energy, arising from compression, one has

$$(\overline{V}_p K) = (\overline{k}'\, r^2 n)\left(\frac{1}{3}\right)^2\left(\frac{\delta V}{V}\right)^2\left(\frac{\overline{V}}{\delta V}\right)^2$$

$$= \left(\frac{\overline{k}'\, r^2\, n}{9}\right)$$

(3.20)

so that upon combining and noting (3.13), one has,

$$(\overline{V}_p\,\bar{\mu}) = (\overline{k}'\, n\, r^2)(\overline{\cos^2\theta})\tfrac{2}{4} + 2 A_c^{1/2}\, r^2\, \mu_{solid}$$

$$= \left(\frac{\overline{k}'\, n\, r^2}{9}\right) + \left(\frac{P}{Y}\right)^{1/2} 4 r^3\, \mu_s$$

(3.21)

since by (3.1), one has

$$\left(A_c^{1/2}\right) = \left(\frac{P}{Y}\right)^{1/2}(2r)$$

(3.22)

for the contact correction term.

Finally, upon dividing, one has the following expression for the skeleton of particles above for shear and compressional stored components of energy

$$\left(\frac{\bar{V}_\rho \bar{\mu}}{\bar{V}_\rho K}\right) = \left(\frac{\bar{\mu}}{K}\right) = \left[\frac{\left(\frac{\overset{45}{\bar{k}'} n r^2}{9}\right) + \left(\frac{\rho}{Y}\right)^{1/2} 4 r^3 \mu_s}{\left(\frac{\bar{k}' n r^2}{9}\right)}\right]$$

$$= \left[1 + \left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36 r}{\bar{k}' N}\right)\mu_s\right]$$

(3.23)

Finally, upon multiplying through by K, one has,

$$\bar{\mu} = K\left[1 + \left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36 r}{\bar{k}' N}\right)\mu_s\right] = \left(\frac{Y}{2}\right)\left(1 + \left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36 r}{\bar{k}' N}\right)\mu_s\right)$$

(3.24)

also,

$$\left(\bar{\lambda} + \frac{2}{3}\bar{\mu}\right) = \bar{K} = \left(\frac{Y}{2}\right) = \left(\bar{\lambda} + \left(\frac{Y}{3}\right)\left(1 + \left(\left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36 r}{\bar{k}' N}\right)\right)\mu_s\right)\right)$$

(3.25)

and,

$$\bar{\lambda} = \left[\left(\frac{1}{2} - \frac{1}{3}\left(1 + \left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36 r}{\bar{k}' N}\right)\mu_s\right)\right]Y$$

Hence, in this development one has obtained the effective Lamé constants $\bar{\lambda}$, $\bar{\mu}$ for the skeleton of particles alone, i.e. particles and vacuum. It is now necessary to pass on to the composite system of the skeleton of particles and the surrounding fluid.

For the composite system, one now has by superposition of the energies of the separate components,

E total (composite) = E_{shear} + $E_{compressional}$ + (E fluid)

$$(3.27)$$

Now this is,

$$E_{total} = \left(\frac{K_{mixture}}{2}\right)\left(\frac{\delta\bar{V}}{\bar{V}}\right)^2 = \frac{1}{2}\,\bar{u}\left(\frac{\delta\bar{V}}{\bar{V}}\right)^2$$
$$+ \frac{1}{2}\,K_{skeleton}\left(\frac{\delta\bar{V}}{\bar{V}}\right)^2 + \frac{1}{2}\left(\frac{\bar{V}_{fluid}}{\bar{V}_{total}}\right)K_{fluid}\left(\frac{\delta\bar{V}}{\bar{V}}\right)^2$$

$$(3.28)$$

which is the shear, compressional, and compressional components of the system energy for the solid and fluid respectively with $K_{mixture}$ representing the incompressibility of the composite system and K_{fluid} the incompressibility of the fluid.

But from prior developments,

$$K_s = \left(\frac{Y}{2}\right)$$

$$(3.29)$$

and,

$$\bar{u} = \left(\frac{Y}{2}\right)\left(1 + \left(\frac{\rho}{Y}\right)^{1/2}\left(\frac{36r}{\bar{\lambda}'\omega}\right)\mu_s\right)$$

$$(3.30)$$

with $\bar{k}'$ being identified with (Y/r).

Now for the composite, one obtains for the incompressibility

$$\tfrac{1}{2} K_{mixture} \left(\frac{\delta \bar{V}}{\bar{V}} \right)^2 = \tfrac{1}{2} K_{skeleton} \left(\frac{\delta \bar{V}}{\bar{V}} \right)^2 + \tfrac{1}{2} K_{fluid} \left(\frac{\delta \bar{V}}{\bar{V}} \right)^2 \left(\frac{\bar{V}_{fluid}}{\bar{V}_{total}} \right)$$

(3.31)

From which obtains, upon division,

$$K_{mixture} = K_{skeleton} + K_{fluid} \left(\frac{\bar{V}_{fluid}}{\bar{V}_{total}} \right)$$

(3.32)

From (3.29), one has,

$$K_{mixture} = \left(Y/2 \right) + K_{fluid} \left(\frac{\bar{V}_{fluid}}{\bar{V}_{total}} \right)$$

(3.33)

By the same arguments as above for the composite in considering shear, since the fluid cannot support shear and does not support shear energy, one has,

$$\tfrac{1}{2} \bar{\mu}_{mixture} \left(e_{xy} \right)^2 = \tfrac{1}{2} \bar{\mu}_{skeleton} \left(e_{xy} \right)^2$$

(3.34)

Therefore

$$\bar{\mu}_{mixture} = \bar{\mu}_{skeleton}$$

$$= \left(Y/2 \right) \left(1 + \left(\rho/Y \right)^{1/2} \left(\frac{36 r}{k N} \right) u_{solid} \right)$$

(3.35)

and,

$$\overline{\lambda}_{mixture} = K_{mixture} - \left(\frac{2}{3}\right) \overline{\mu}_{mixture}$$

(3.36)

Hence one has arrived at the microscopically derived effective
Lamé parameters, $\overline{\lambda}$, $\overline{\mu}$ for the particles surrounded by
fluid or vacuum.

(b) Velocity Calculations by the Theory and Comparison with Data for Sand

It is now of value to implement the microscopic theory
for generation of some predicted results and comparison
of these with experimental data. The velocities obtained
will serve as a prediction, and the general description
of the wavefield may subsequently be obtained by substituting
in $\overline{\lambda}$, $\overline{\mu}$ for the various environments into the general
solution to yield fields that are calculable.

One begins the calculations by taking the averaged
values of the parameters involved as

$$\rho = 3 \, gm/cc \; ; \quad \gamma = 2000 \, atm \; ; \quad P = 3 \, atm \; (1 \, atm - gas\,case)$$

$$r = .1 \, cm \; ; \quad \mu_{solid} = 10^{11} \, erg/cc \; ; \quad \overline{k}' = 2000 \, atm$$

Upon substituting these values into the equations for
the velocities for the microscopically determined parameters,
$\overline{\lambda}$, $\overline{\mu}$, there is obtained:

Vacuum and Particles

$$\bar{V}_{shear} = \left(\frac{\bar{u}}{\ell}\right)^{1/2} = \left[\frac{Y/2 + Y/2 \left(P/Y\right)^{1/2} \left(\frac{36r}{kN}\right) \mu_S}{\ell}\right]^{1/2}$$

$$\bar{V}_{shear} = 200 \, m/sec \tag{3.37}$$

and,

$$\bar{V}_{compressional} = \left[\frac{k + 4/3 \, \bar{u}}{\ell}\right]^{1/2} ;$$

$$\bar{V}_{compressional} = \left[\frac{Y + (Y/2)\left(P/Y\right)^{1/2}\left(36r/kN\right)\mu_S}{\ell}\right]^{1/2} = 280 \, m/sec \tag{3.38}$$

Liquid and Particles

Here, one has,

$$K_{mixture} = K_{skeleton} + \left(\frac{\bar{V}_{liq}}{\bar{V}_{total}}\right) K_{liquid}$$

$$\left[\bar{V}_{liq} / \bar{V}_{total}\right] = .3 \tag{3.39}$$

for

$$K_{mixture} = (.7)(2000) + (.3)(20000) = 7400 \, atm$$

also,

$$\bar{u}_{mixture} = \bar{u}_{skeleton}$$

$$\tag{3.40}$$

so that

$$\bar{v}_{shear} = \left[\frac{\mu}{\rho}\right]^{1/2} = 200 \, m/sec$$

(3.41)

and

$$\bar{v}_{compressional} = \left[\frac{k + 4/3\,\bar{\mu}}{\rho}\right]^{1/2} = 530 \, m/sec$$

(3.42)

Gas and Particles Case

Here,

$$K_{mixture} = K_{skeleton} + (small \; order \; terms)$$

$$\simeq K_{skeleton}$$

(3.43)

This is a small order correction but for a gas at one atmosphere it is only one part in one thousand.

And also,

$$\bar{\mu}_{mixture} = \bar{\mu}_{skeleton} + (small \, order \, term) \simeq \bar{\mu}_{skeleton}$$

so that one has,

$$\bar{V}_{gas} = \bar{V}_{vacuum} + \epsilon \sim 200\, m/sec$$

$$\bar{V}_{compressional} = \bar{V}_{vac_{compressional}} + \epsilon \sim 280\, m/sec$$

(3.44)

As a result of the foregoing, one may construct the following table.

TABLE III

Calculated Velocities vs Experimental Data	
Model	$\bar{V}$ Compressional
Vacuum Case	280 m/sec
Liquid Case	530 m/sec
Gaseous Case	280 m/sec
Experimental	$\bar{V}$ Compressional
Vacuum Case	205-300 m/sec
Liquid Case	350 m/sec
Gaseous Case	300 m/sec

Where in the experimental data the vacuum case included

sand beds with wave velocities measured by Eden and Felsenthal,

and Antsyferov; the liquid case included 1% gas and sand

and the wave velocity was measured by Brandt, while the

gaseous case involved sand and gas in the interstices

and was measured by Gassman. The range of agreement is

good for the microscopic effective parameters.

(c) <u>Development of the Energy Loss Mechanisms for the Powder</u>

The generalized equation of motion in the inhomogeneous

lossy medium (here the powder) is a nonlinear partial

differential equation. At this point, one proceeds to

develop the specific components for the loss function

L for the powder in order to obtain the nonlinear partial

differential equation as it specifically relates to the

powder. This will be a form of the generalized wave equation

for an inhomogeneous, lossy, granular medium. The equation

has been solved for the general case and its near and

farfield solutions have been examined over the frequency

range. For the actual powder, a solution may be obtained.

Symbolically, L is made up of the following components,

$$ L = S + T + V + F + A + B $$

$$ (3.45) $$

Where, recalling the definitions of Chapter II, S represents

scattered energy out from the primary wave due to in homo-
geneities, F is the frictional loss arising from intergranular
sliding. The term V represents viscous energy dissipated
into the fluid by the motion of the granules relative
to the fluid, while the term T represents the thermodynamic
loss (least conduction). The terms A and B represent
intrinsic attenuation in the solid and the liquid respectively.
One now proceeds to calculate the form for L.

As a preliminary to determining the losses in the
powder it is necessary to examine the motion and stresses
on the granules. The starting point is to consider a
granule in static equilibrium under a pressure, Peq.
One has, for Ag, surface area of granule, A_c, contact
area, Y the yield stress and n contact points,

$$\left(P_{eq}\, A_g\right) = Y A_c \left(\sum_{n=1}^{n/2} n \cos\theta \right)$$

$$(3.46)$$

Summing over all angles θ of intersection, for the average
value of cos θ, one obtains,

$$\left(\frac{n}{2}\right)(Y)\left(\frac{A_c}{A_g}\right)\left(\frac{1}{\sqrt{3}}\right) = P_{eq} = \left(\frac{n\, A_c\, Y}{2\, A_g\, \sqrt{3}}\right)$$

$$(3.47)$$

Where the term $\left[\sum_{n=1}^{n/2} n \cos\theta \right]$ arises since the stresses
act over half the contact points on either side of the
granule. This is a balanced force condition.

Now added to this static pressure Peq one superposes
the oscillating pressure induced in the dynamic case from
the motion of the passing wave. Since the granules will
not depart significantly from their equilibrium positions,
then one employs the harmonic approximation, whereby the
first nonvanishing correction to the equilibrium potential
energy is given by quadratic terms in the expansion of
the potential energy U pot about its equilibrium value.
In this form one has:

$$U_{pot} = U^{eq} + U^{harmonic}$$

$$(3.48)$$

The harmonic potential energy is expanded in terms of
the oscillation displacements u(na), u((n+1)a) for granules
that oscillate about equilibrium positions x = na, x =
(n+1)a... due to the wave motion. The deviation in the
potential energy, $U^{harmonic}$ goes on the square of the
displacement, hence the following form is obtained.

$$U^{harmonic} = \left(\frac{K'}{2}\right)\left(\sum_{n} u(na) - u((n+1)a)\right)^{2}$$

$$(3.49)$$

Here, K' is an effective spring constant depending on
the elastic properties of the granules since energy in

a spring lattice goes as x^2 for x the displacement. In this case, a grid network has been superposed about the granules for a reference system and consists of equilibrium points a, 2a,...na.

Formulation of the equations of motion for displacements about the equilibrium positions in terms of finite difference equations, yields

$$M_i \ \ddot{u}(na) = -\frac{\partial U}{\partial u(na)} = -\left[(2) u(n) - 2u(n+1) + u(n+1) \atop -u(n-1)\right]$$

$$M_i \ \ddot{u}(na) = -(2 \ u(n) - u(n+1) - u(n-1)) K'$$

$$(3.50)$$

This set of equations of motion is the same as that of a set of masses connected by spring constant K'. The solutions to these equations of motion are developed from the auxiliary equation for (3.50)

$$U(x,t) = u(na,t) = A e^{i(kx-wt)} \underset{=}{} A e^{i(k(na)-wt)}$$

$$(3.51)$$

Substitution of (3.51) into the equations of motion yields,

$$\omega^2 M_i e^{i(kna-\omega t)} = -K'\left(2e^{i(kna-\omega t)} - e^{i(k(n-1)a-\omega t)} - e^{i(k(n+1)a-\omega t)}\right)$$

(3.52)

so that one has,

$$-\omega^2 M_i e^{i(kna-\omega t)} = -K'\left(2 - e^{-ika} - e^{ika}\right)\left(e^{i(kna-\omega t)}\right)$$

$$= -2K'\left(1 - \left(\frac{e^{-ika} - e^{ika}}{2}\right)\right)e^{i(kna-\omega t)}$$

$$= -2K'\left(1 - \cos(ka)\right)e^{i(kna-\omega t)}$$

(3.53)

so that

$$\omega^2 = \left(\frac{K'}{M_i}\right)\left(1 - \cos(ka)\right)$$

(3.54)

The above equation represents the dispersion relationship for no losses for the particle chain. Here Mi is the mass of the ith granule and a is the distance between

granules. As a result of this, the real displacement
with the particle chain takes the form (for no losses):

$$U(x,t) = \cos(kna - wt)\,A$$
$$= \cos(kx - wt)\,A \qquad (3.55)$$

Now in this system, the force is proportional to the net
displacement so, from this net displacement ∇U, one has,

$$P_{oscillating} = \left(\frac{-f'}{A_g}\right) = \left(\frac{C\nabla U}{A_g}\right) = \left[\frac{K'(U_1 - U_2)}{A_g}\right]$$

$$(3.56)$$

In view of (3.55) and (3.56) one has,

$$P_{oscillating} = \left(\frac{-K'}{A_g}\right) k \sin(kx - wt)$$

$$(3.57)$$

Now the elastic or equilibrium pressure acts to retard
the effect of the oscillating pressure, .i.e., the amplitude
of the oscillating pressure must exceed the effective
frictional pressure arising from the static pressure,
before any sliding motion can occur. In this regard,
it is possible to formulate the net driving pressure for
granules and vacuum as:

$$P_{total} = \left|\left(\frac{-K'}{A_g}\right)\left(k\sin(kx - wt) + \mu Y \left(\frac{A_c}{A_g}\right)\left(\sum_{n=1}^{n/2} n\cos\theta\right)\right)\right|$$

$$(3.58)$$

Now for the case of granules in fluids (gas, liquid),
the net pressure will now have a retarding pressure opposing
it due to the viscosity of the fluid. Viscous forces,
f' generally may be described as:

$$f' \propto \text{velocity gradient}$$

(3.59)

$$f' = (\eta A_g) b (\Delta v)$$

(3.60)

So that,

$$P_{viscous} = (\eta b \Delta v)$$

(3.61)

for b a constant inversely proportional to spacing between
particles, Ag the exposed area of the particle, and ΔV
the relative motion of the fluid with respect to the particle.

At this point, now one has first the background equili-
brium pressure in the elastic case, and, superimposed
on that, the dynamic pressure brought about by the passing
of the wave through the medium. To this, one must add
the viscous damping pressure that opposes the driving
force.

The result becomes, upon superposition,

$$P_{total} = \left| P_{osc} - P_{viscous} - P_{hydrostatic} \right|$$

$$(3.62)$$

$$P_{total} = \left| \left(\frac{-K'}{Ag} \right) k \sin(kx - wt) + \mu Y \left(\frac{Ac}{Ag} \right) \left(\sum_{n=1}^{n/2} n \cos\theta \right) + \eta \, b\,w \, k \sin(kx - wt) \right|$$

$$(3.63)$$

but as will be seen, one can express ΔV as

$$\Delta V = w \, \Delta U$$

$$(3.64)$$

in the real parts, where ΔV represents the motion of fluid
with respect to particles and ΔU is the relative displacement
of particles and fluid. In view of (3.64) and (3.63),
one has,

$$P_{total} = \left| \mu Y \left(\frac{Ac}{Ag} \right) \left(\sum_{n=1}^{n/2} n \cos\theta \right) + k \sin(kx - wt) \left[\eta b w - \frac{K'}{Ag} \right] \right|$$

$$(3.65)$$

When η is the viscosity of the fluid and the other parameters
have been defined earlier.   There considerations will
be useful in the formulation of the loss mechanisms for
the powder and the handling of the propagation of elastic
waves in the powder.

(1)   Frictional Loss

One is now in a position to calculate the specific
form for L.   One obtains the following for F, the dissipated
energy due to frictional rubbing of the granules against
each other one obtains,

$$F = \left(\frac{w}{V}\right)\left(\sum_{j} (ka)\mu \int_{R_o}^{R} (\text{Hydrostatic } A_i) \cdot dR_j\right)$$

$\cdot$ (3.66)

$$F = \left(\frac{w}{V}\right)\left(\sum_{j} (ka)\mu \int_{t_o}^{t} (P_{hyd} A_i)\left(\frac{\partial u_{ji}}{\partial \tau}\right) dt\right)$$

(3.67)

with the condition,

$$f'_{net} = \left(\nabla (P_{net} A) r - f'_{viscous}\right)$$

(3.68)

with t_0 given by the point, where in the wave cycle,

$$f'_{net} \geq \mu \, P_{hyd} \, A$$

(3.69)

Here P_{net} is the net effective pressure (after overcoming
the hydrostatic pressure Y ($\sum_{n=1}^{n/2} nc\alpha\theta$) that prevents
sliding). The integral goes over the time from t_0 to
t in the cycle that motion occurs while Ai is the cross
sectional area of the ith granule, μ' the coefficient
of friction between granules and $(k \alpha \frac{\partial w_{ij}}{\partial \tau})$ the relative
velocity of the ith granule over the jth granule, while
$(\omega/\bar{V})$ accounts for the loss on a per unit volume per unit
time basis, for ω the angular frequency and $\bar{V}$ the volume.

One now passes on to the specific forms for the fric-
tional loss for granule and vacuum, granules and liquid
and granules and gas. The functional loss for the liquid
case takes the form:

$$F_{liquid} = \left(\frac{\omega}{\bar{V}}\right)\left(\sum_{j}(k\alpha)\mu \int_{t_0}^{t} \omega Y \left(\frac{A_c}{A_g}\right)\left(\sum_{n=1}^{n/2} nc\alpha\theta\right)(A_i dt)(\sin(kx-\omega t)\right.$$

(3.70)

with the t_0 time point given by

$$\mu \, P_{hyd} \, A \leq f'_{net} = \left| -k^2 K' c\alpha (kx-\omega t) + \eta \sin(kx-\omega t)\omega t \right.$$

(3.71)

Where the additional term in f'net acts as a type of Stokes loss. Similarly, for gas and particles, one has,

$$F_{gas} = \left(\frac{W}{V}\right)\left(\sum_{j}(ka)\mu\int_{t_o}^{t}\omega Y\left(\frac{Ac}{Ag}\right)\left(\sum_{n=1}^{h/2}(n\cos\theta)(Ag\,dt)\sin(kx-\right.$$

(3.72)

with the t_o time point given by

$$\mu P_{hyd}A \leq f'net = \left| -k^2 K'\cos(kx-\omega t) + \eta_g\,\omega Ag\sin(kx-\omega t)\right|$$

(3.73)

where η_g is the viscosity of the gas.

Lastly, for the frictional loss F_{vac} for vacuum and particles, there arises,

$$F_{vac} = \left(\frac{W}{V}\right)\left(\sum_{j}(ka)\mu\int_{t_o}^{t}Y\left(\frac{Ac}{Ag}\right)\omega\left(\sum_{n=1}^{h/2}n\cos\theta(Ag)\,dt\sin(kx-\omega t)\right)\right)$$

(3.74)

with the t_o time point given by

$$\left(\mu P_{hyd}A\right) \leq f'net = \left| -K'k^2\cos(kx-\omega t)\right|$$

(3.75)

Here, $\eta = 0$ and the viscous Stokes loss does not appear. From the above equations one sees that the integration point t_o depend upon the medium.

(2) Viscous Loss

At this point, one passes on to the development of
the viscous loss. One develops V, the viscous energy
dissipated into the fluid by the motion of the skeleton
of granule relative to the fluid.

To obtain V, one considers that superimposed on the
average velocity field of the fluid one has the fluctuations
$\delta \dot{U}$ so that the total velocity $\dot{U}_{tot}$ is composed of an average
velocity $\dot{U}$ and fluctuations $\delta \dot{U}$.

$$\dot{U}_{total} = \bar{U} + \delta \dot{U}$$

(3.76)

and, the velocity gradient becomes,

$$\nabla \dot{U} = k \omega U + \left(\frac{\delta \dot{U}}{r}\right)$$

(3.77)

where r is the granule radius since the channel widths
are of the same order as the granule dimensions. The
first term can be neglected. The velocity gradient is
related to pressure fluctuations ΔP that lead to viscous
loss by relative motion; consequently, one has for the
rate of change of viscous energy with time, (rate of energy
extraction in the composite medium on a per unit volume
basis),

$$-\left(\frac{dE}{dt}\right)_{viscous} = \eta \left(\frac{A}{V}\right)\left(\frac{dS}{dt}\right)\left(\frac{\Delta \dot{U}}{r}\right) - \eta \left(\frac{\Delta \dot{U}}{r}\right)^2$$

(3.78)

Where it is noted that one has two velocity fields, $\dot{U}_1$ and $\dot{U}_2$ in the solid and in the fluid environment respectively. The relative velocity $\Delta \dot{U}$ is defined as:

$$\Delta \dot{U} = \left| \dot{U}_1 - \dot{U}_2 \right|$$

(3.79)

Then, from (3.78) one has,

$$-\left(\frac{dE}{dt}\right)_{viscous} = \eta \left(\frac{\Delta \dot{U}}{r}\right)^2$$

(3.80)

But,

$$P = \left(\overline{\lambda} \, k \, U\right)$$

(3.81)

for the pressure, and

$$E = \left(\overline{\lambda} \, k^2 \, U^2\right)$$

(3.82)

for the energy.

The units of ΔP are in terms of energy per unit volume. Consequently, one has,

$$-\left(\frac{dE}{dt}\right)_{viscous} = \text{decay of turbulence due to viscous loss}$$

$$(3.82)$$

and,

$$-\left(\frac{dE}{dt}\right)_{viscous} \leq (\omega \, \Delta p)$$

$$(3.83)$$

as an upper limit on viscous loss, since the total amount of energy dissipated can be no greater than the total energy. From (3.81) one has,

$$\left(\frac{\Delta p}{p}\right) = \frac{\Delta\bar{\lambda}}{\bar{\lambda}}$$

$$(3.84)$$

When averaged over the cycle that the decay rate occurs, one has,

$$\left(\frac{-dE}{E\,dt}\right)_{viscous} (\omega\tau) \leq \left(\frac{\omega^2 \, \tau \, \Delta p}{E}\right)$$

$$= \left(\frac{\Delta\bar{\lambda}}{\bar{\lambda}} \quad p\left(\frac{\omega^2 \, \tau}{E}\right)\right)$$

$$= \left(\frac{\Delta\bar{\lambda}}{\bar{\lambda}}\right)\left(\frac{\omega^2 \, \tau}{k \, U}\right)$$

$$(3.85)$$

With E the energy of fluid flow. But from (3.80),
with

$$E = \rho \ (\Delta \dot{v})^2$$

<div align="right">(3.86)</div>

and from the relaxation time approximation, one has,

$$\left(\frac{1}{\tau}\right) = \left(\frac{-dE}{E\,dt}\right) = \left(\frac{\eta}{\rho\,r^2}\right)$$

<div align="right">(3.87)</div>

So that, from (3.85) and the above, one has,

$$\left(-\frac{1}{E}\frac{dE}{dt}\right)_{viscous}(\omega\,\tau) = \left(\frac{\omega^2}{k\,v}\right)\left(\frac{\rho\,r^2}{\eta}\right)\left(\frac{\Delta\bar{\lambda}}{\bar{\lambda}}\right)$$

<div align="right">(3.88)</div>

Equation (3.88) is the viscous energy dissipated per unit
volume per unit time, and is the averaged rate of energy loss
over many cycles. For the vacuum case, these terms
do not appear since there is no medium to dissipate viscous
energy.

(3) Intrinsic Losses in the Solid
and the Liquid

The term A is included in the general (dE/dt) expression
to account for intrinsic attenuation in the solid granules,
whereas the term B is included in the general loss expression
to account for intrinsic attenuation in the liquid. From

the Akhieser theory for phonon rate processes[5] one has, for the solid granules,

$$A = \left(\frac{-dE}{E\,dt} \right)_{solid} = \left(\frac{1}{\lambda_{solid}} \right) \left(C_{v\,solid}\,T\,(\gamma - \bar{\gamma})^2 \right) \omega^2 \tau$$

$$(3.39)$$

where,

$$\gamma = \left(\frac{1}{\bar{T}} \left(\frac{d\bar{T}}{d\epsilon} \right) \right)$$

$$(3.90)$$

where $\bar{T}$ is the temperature and ϵ the strain, and, also,

τ = phonon relaxation time,

C_v = specific heat at constant volume

ω = frequency of the wave

$\bar{\lambda}$ = Lamé parameter

$\bar{\gamma}$ = average γ

In addition one has for B, the intrinsic attenuation in the liquid, by a similar argument as advanced in Appendix 5, the following expression:

$$B = \left(\frac{-dE}{E\,dt} \right)_{liquid} = \left(\frac{1}{\lambda} \right)_{liquid} \left(C_{v\,liquid}\,T\,(\gamma - \bar{\gamma})^2 \right) \omega^2 \tau$$

$$(3.91)$$

[5]A general derivation of phonon rate processes is discussed in Appendix 5.

The total intrinsic attenuation is additively composed
of (A+B) arising from the contributions of each of the
components is the composite medium. These losses are
in addition to mechanisms operating outside of the granules
also being examined.

(4) Scattering Loss

The scattering loss S for the inhomogeneous granular
medium (powder) is now examined. In order to evaluate
the scattering loss S, it is necessary to formulate the
perturbation Hamiltonian governing the scattering rate
process due to perturbations in the velocity.[6] For the
unperturbed Hamiltonian, H^o, one has,

$$H^o = \sum_r \sum_{r'} \left(\rho v^2 \right) \left(q \, q' \right) e^{i \left(q - q' \right) \frac{\left(n r - n' r' \right)}{\left(a \, a^* \right)}}$$

(3.92)

with the terms n and n' defined as (for granule site weighting),

$$n = \left(\frac{\overline{V_n}}{\overline{V}_o \, _{Ave}} \right)$$

(3.93)

[6] A general discussion of the basic perturbation theory
technique of which this is a generalization, may be found
in such books as J. Ziman, Principles of the Theory of
Solids (Cambridge: Cambridge University Press, 1964.)

$$h' = \left(\overline{V}_{n'} / \overline{V}_0 \; ae \right)$$

(3.94)

and ρ is the density of an average medium (q,q') are the body wave vectors for incident and scattered waves respectively, and r and r' are the locations of the scattering source and receiving points respectively. The terms (aa^*) includes the phonon creation and destruction operators for scattering and are proportional to energy density. Now,

$$H = H^o + H'$$

(3.95)

Where H^o is the unperturbed Hamiltonian, H' the perturbed Hamiltonian, and H the total Hamiltonian representing the energy. Formulating H explicitly yields,

$$H = \sum_r \sum_{r'} \left(\rho (v + n\Delta v + n'\Delta v')^2 (qq')(aa^*)\left(e^{i(q-q')(nr-n'r')} \right) \right)$$

(3.96)

Expanding H one has,

$$H = \sum_r \sum_{r'} \rho (v^2 + vn\Delta v + n'\Delta v\, v' + h.o.t.) e^{i(q-q')(nr-n'r')}_{(qq')(aa^*)}$$

(3.97)

i.e.,

$$H = \sum_r \sum_{r'} \left(\rho v^2 + (\rho n v \Delta v) + \rho (n'\Delta v' v) + (h.o.t.) \right)(qq') e^{i(q-q')(nr-n'r}_{(aa^*))}$$

(3.98)

Upon extracting the perturbation Hamiltonian H' and truncating the higher order terms, one has,

$$H' = \sum_r \sum_{r'} \left(\ell n (v \Delta v) + \ell n' (v \Delta v') \right) \left(gg' \right) e^{i(g-g')(nr-n'r')} \frac{}{(\alpha \alpha^*)}$$

(3.99)

Now in general in elastic limits, for J and C constants

$$\bar{v}^2 = \left(\frac{J \bar{\mu} + C \bar{\lambda}}{\ell} \right)$$

(3.100)

so that

$$\ell = \left(\frac{J \bar{\mu} + C \bar{\lambda}}{\bar{v}^2} \right)$$

(3.101)

So that one may arive at the following for the perturbation Hamiltonian H' for scattering due to a perturbation in velocity induced by a perturbation in the medium:

$$H' = \sum_r \sum_{r'} \left(J \bar{\mu} + C \bar{\lambda} \right) \left(\frac{n \Delta v}{v} + \frac{n' \Delta v'}{v} \right) \left(gg' \right) e^{i(g-g')(nr-n'r')} \frac{}{(\alpha \alpha^*)}$$

(3.102)

Now the displacement of a point r due to the superposition of elastic waves of given wave vector, polarization j, having attenuation can be represented in the following manner.

$$U(r) = \left(\frac{1}{\sqrt{G}}\right) \sum_{q} \epsilon_j (q_1 + i q_2) \, v \, a(q_1 + i q_2) \, e^{i\omega t}$$

(3.103)

i.e.,

$$U(r) = \left(\frac{1}{\sqrt{G}}\right) \sum_{q} \epsilon_j (q_1 + i q_2) e^{i q_1 \cdot nr} (e^{-q_2 nr}) e^{i\omega t} a(q_1 + i q_2)$$

(3.104)

It is understood that in the total unperturbed Hamiltonian
the general summation goes over all modes for incident
body waves q and scattered body waves q' as well as incident
surface waves $\bar{k}$ and scattered surface waves $\bar{k}'$. Thus
the total unperturbed Hamiltonian apppears as:

$$H^0 = \sum_{qq' jj'} H^0(q,j) = \sum_{q,j} (\rho v^2)(a a^*)(q q') e^{i(q-q')(nr-n'r')}$$

(3.105)

The perturbation Hamiltonian then becomes,

$$H' = \left(\frac{\bar{\mu}}{G}\right) \sum_{qq'} \sum_{j,j'} (n\Delta v + n'\Delta v')(a_q a_{q'})(\epsilon \cdot \epsilon')(q q') \cdot$$ (3.106)
$$\cdot \, e^{-i(\omega+\omega')t}(e^{i(q_1-q_1')nr}) e^{-(q_2-q_2')(nr)}$$

Eq. (3.106) represents the perturbation Hamiltonian for
excess attenuation and scattering of elastic waves of
mode (q,j) being scattered into mode (q',j') in the inhomo-
geneous granular medium. It can be represented functionally
as:

$$H' = \sum_{gg'} \sum_{jj'} C_{gg'jj'} \, a^*(g,j) \, a(g,j)$$

<div align="right">(3.107)</div>

while in the double sum, one has body waves and surface
waves of the form

$$C_{gg'} (a^*(g))(a(g))$$

and

$$C_{\bar{k}'g} \, (a(g))(a^*(\bar{k}'))$$

where the coefficients C are defined in the form:

$$C_{gg'} = \left(\frac{\bar{u}}{G}\right) \sum_{gg'} (n\Delta v + n'\Delta v')(\epsilon_{q'}^* \cdot \epsilon_q)(gg')(e^{-i(w+w')t}) \cdot$$
$$\cdot (e^{i(g,q' + iq_z - iq_z')nr})$$

<div align="right">(3.108)</div>

which becomes,

$$C_{gg'} = \left(\frac{\bar{u}}{G}\right) \sum_{gg'} (n\Delta v + n'\Delta v') e^{i(g_1 - g_1')(nr)} (gg') \cdot$$
$$\cdot (\epsilon_{q'}^* \cdot \epsilon_q)(e^{-(q_z - q_z')nr})(e^{-i(w+w')t})$$

<div align="right">(3.109)</div>

also,

$$C_{q\bar{k}'} = \left(\frac{\bar{\mu}}{G}\right)\left(\sum_{\bar{k}'q}(n\Delta v + n'\Delta v')e^{i(q_1 - \bar{k}_1')nr}(q\bar{k}')\cdot\right.$$
$$\left.\cdot(\epsilon_q \cdot \epsilon_{\bar{k}'}^*)e^{-(q_2 - \bar{k}_2')(nr)}e^{-i(\omega + \omega')t}\right.$$

(3.110)

so, finding the absolute modulus of the coefficients yields:

$$\left|C_{q\bar{k}'}\right|^2 = \left(\frac{\bar{\mu}}{G}\right)^2\left(\sum_{rr'}(n\Delta v + n'\Delta v')^2\left|\epsilon_q \cdot \epsilon_{\bar{k}'}\right|^2(q k')^2\cdot\right.$$
$$\left.\cdot\left(e^{i(q_1 - \bar{k}_1')(nr - n'r')}\right)\left(e^{-(q_2 - \bar{k}_2')(nr - n'r')}\right)\right.$$

(3.111)

and also,

$$\left|C_{qq'}\right|^2 = \left(\frac{\bar{\mu}}{G}\right)\sum_{rr'}(n\Delta v + n'\Delta v')^2(qq')\left|\epsilon_q'^* \cdot \epsilon_q\right|\cdot$$
$$\cdot e^{i(q_1 - q_1')(nr - n'r')}e^{-(q_2 - q_2')(nr - n'r')}$$

(3.112)

The double sum over pairs of granules in the perturbation
Hamiltonian lead to various interference terms in the
scattering by the arrays of granules. To consider the
scattering of energy from a body wave q, one must consider
other interacting surface wave states k' and all body
waves q'. This wave vector q can be longitudinal

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or transverse (P or S). The total relaxation time for
the scattering processes is additively composed of those
for the separate processes:

$$\left(\frac{1}{\tau}\right) = \left(\frac{1}{\tau_{surface\ wave}}\right) + \left(\frac{1}{\tau_{Bulk\ wave}}\right)$$

(3.113)

One deduces the rate of change of $E(q,j)$ the energy content
of mode (q,j) as a result of scattering through the relaxa-
tion time formulation:

$$\left(t\frac{dE}{dt}\right) = \left(-\frac{tE}{\tau}\right) = \sum_{q'j} \left(\frac{2|c|^2}{m^2\omega\omega'}\right)\left(\frac{1-\cos\Delta\omega t}{\Delta\omega^2}\right)\left(E-E'\right)$$

(3.114)

Where the factor $\left(\frac{1-\cos\Delta\omega t}{\Delta\omega^2}\right)$ is a resonance factor that
selectively picks out the appropriate scattered and incident
modes in the summation. The frequencies ω and ω' between
incident and scattered modes obey conservation of frequency.
Now E' vanishes for all modes of q' except the scattered
mode. From Eq. (3.114) one has,

$$\left(-\frac{t}{\tau}\right) = \sum_{q'j} \left(\frac{2|c|^2}{m^2\omega\omega'}\right)\left(\frac{1-\cos\Delta\omega t}{\Delta\omega^2}\right)$$

(3.115)

Now the summation over q' scattered body waves is associated
with the form

$$\sum_{q} \longrightarrow \left(\frac{\mathcal{G}a^3}{(2\pi)^3}\right)\int d\omega' \int \frac{dS'}{\bar{v}}$$

(3.116)

for incident surface and bulk waves.

With

$$\bar{V}' = \left[\frac{d\omega}{dq'}\right]$$

(3.117)

and G is the number of granules per unit volume, a^3 the volume of the particles, and dS' represents a surface element of constant $\omega = \omega'$ in q' space and where the major contribution comes from $\Delta\omega = 0$ ($\omega = \omega'$).

From (3.116) one has,

$$\left(\frac{1}{\tau_{Bulk}}\right) = \left(\frac{Ga^3}{(2\pi)^3}\right)\left(\frac{1}{m^2\omega^2}\right)\sum_j \frac{dS'}{|\bar{V}'|}\,|C_{qq'}|^2$$

(3.118)

for m the mass of the granules.

By a similar train of thought for scattering into surface waves, one has the association,

$$\sum_{k'} \longrightarrow \left(\frac{Ga^2}{(2\pi)^2}\right)\int d\omega'\left(\frac{dL'}{\bar{V}'}\right)$$

(3.119)

with dL' a line element of constant $\omega = \omega'$ in k' space.

As a result of (3.118) one has,

$$\left(\frac{1}{\tau_{Bulk}}\right) = \frac{Ga^3}{(2\pi)^2(m^2\omega^2)}\left[\sum_{\epsilon,\epsilon^*}\int \frac{dS}{|\bar{V}_j|}\,|C_{qq'}|^2\right]$$

(3.120)

while from (3.119) one has

$$\left(\frac{1}{\tau_{surf}}\right) = \frac{G\,a^3}{(2\pi)(m^2\omega^2)} \oint \frac{\left|C_q\,\bar{k}'\right|^2\,dL'}{\left|\bar{V}_j\right|}$$

(3.121)

where dL' is an element of line of constant $\omega = \omega'$ in

k' space and dS' is an element of surface of constant

$\omega = \omega'$ in q' space. These elements take the form:

$$dL' = k'\,d\phi'$$

(3.122)

and

$$dS' = q'^2\,\sin\theta\,d\theta\,d\phi$$

(3.123)

Thus the partial scattering rates between surface and

bulk (shear and compressional) waves with excess attenuation

is seen in the following equations:

$$\left(\frac{1}{\tau_{surf-wave}}\right) = \left[\int_0^{2\pi} \sum_{k'} d\phi' \sum n(\Delta v)(n'\Delta v')\left|\epsilon_{\bar{k}'}^*\,\epsilon_q\right|^2 \cdot \right.$$

$$\cdot \left(e^{-(q_z-\bar{k}_z')(nr-n'r')}\right)(q\,\bar{k}')^2\left(\frac{2\mu^2 a^2}{\pi m^2 \omega^2\,\tau_{surf}\,G}\right)\cdot$$

$$\left. \cdot\,e^{i(q_t-k_t')(nr-n'r')} \right]$$

(3.124)

and similarly for bulk waves,

$$\left(\frac{1}{\tau_{Bulk}}\right) = \left[\int_0^{2\pi} \sum_{q'} d\phi' \sum_{rr'} (n\Delta v)(n'\Delta v') / \epsilon_q^* \epsilon_q \right| \cdot$$

$$\cdot \left[\frac{u^2 a^3}{6\pi^2 m^2 \omega^2 \bar{v}_{bulk}} \right] (qq')^2 \left(e^{i(q_1-q_1')(nr-n'r')} \right)$$

$$\cdot \left(e^{-(q_2-q_2')(nr-n'r')} \right)$$

$$(3.125)$$

The integrals from which (3.124) and (3.125) have arisen have been over lines and surfaces of constant $\omega = \omega'$ which is a statement of those processes that conserve energy which is the condition under which the two components of interaction in the scattering can exchange energy between themselves. Finally, from the general description of the relaxation time τ, one has,

$$\left(\frac{dE}{dt}\right) = -\left(\frac{E}{\tau}\right) = -E\left(\frac{1}{\tau_{Bulk}} + \frac{1}{\tau_{surface}}\right)$$

$$(3.126)$$

for the total time rate of change of energy for mode (q,j). This energy disappears as a result of scattering with absorption. Hence, for the scattering loss term one has,

$$S = -\left(\frac{1}{E}\frac{dE}{dt}\right) = \frac{1}{\tau} = \left(\frac{1}{\tau_{Bulk}} + \frac{1}{\tau_{surface}}\right)$$

$$(3.127)$$

Here, using (3.127), (3.125) and (3.124) one obtains,

$$S = \left(-\frac{dE}{E\,dt}\right) = \left(\frac{2\bar{u}^2 a^2}{\pi m^2 \omega^2 \bar{V}_{surf}\,G}\right) \cdot$$

$$\cdot \left[\int_0^{2\pi} d\phi' \sum_{k'}\sum_{rr'} (n\Delta v)\,\omega(n'\Delta v')\,\left|\,\epsilon_{\vec{k}'}\cdot\epsilon_{\vec{q}}\,\right|^2 (\vec{q}\cdot\vec{k}')\right. :$$

$$\left. \cdot e^{i(\vec{q}_1 - \vec{k}_1')(nr - n'r')}\left(e^{-(\vec{q}_2 - \vec{k}_2')(nr - n'r')}\right)\right]$$

(3.128)

(surface wave relaxation time)

$$+$$

$$\left(\frac{\bar{u}^2 a^3}{G\,\pi^2 m^2 \omega^2 \bar{V}_{bulk}}\right)\left[\int_0^{2\pi}\sum_{\vec{q}'} d\phi'\left(\sum_{rr'}(\Delta v)(\Delta v')\,e^{-(\vec{q}_2 - \vec{q}_2')(nr - n'r')}\right.\right.$$

$$\left.\left. \cdot (nn')\,\left|\,\epsilon_{\vec{q}}^*\cdot\epsilon_{\vec{q}}\,\right|^2 \sum_{\vec{q}}(\vec{q}\cdot\vec{q}')^2\,e^{i(\vec{q}_1 - \vec{q}_1')(nr - n'r')}\right]\right.$$

(3.129)

(bulk wave relaxation time)

This is the time rate of change of energy being taken
out of the primary wave (and being transformed into secondary
waves of bulk and surface types) with excess absorption
of the wave occuring between scatterers. The extraction
mechanism for the energy is the scattering mechanism.
In this way the primary wave receives an attenuation.
The scatterers are a result of the mechanical variation
of the medium, and, consequently, the scattered waves
are results of the inhomogeneity of the medium. Each
time the wave scatters energy is removed from the primary

beam (except of course for subsequently scattered energy
being replaced back into the primary beam, i.e., back
scattering). In the scattering process, dispersion takes
place as the scattering acts as a loss mechanism and losses
induce frequency shifts in the wave field. Dispersion
effects are seen as a result of the mechanical inhomogeneity
of the medium. Since low frequency waves are large in
relationship to the size of the scatterers, dispersion
effects due to the variations of the medium are small
since there is not a significant variation over one wavelength,
and the losses are low and the frequency shifts small.
As one advances higher in frequency and the wavelength
shortens, the scatterers play a more dominant role in
the dispersion of the wave.

One has viewed the perturbation as one of velocity
in the medium as a result of a perturbation in the mechanical
properties of the medium. The mechanical variation of
the medium is what lies at the base of the scattering
loss mechanism.

(5) Thermodynamic Loss

At this point, one considers the energy loss due
to heat conduction for the wave propagation process, and
develops the thermodynamic loss. When a strain wave propa-
gates through the material, each region is alternatingly
expanded or compressed. These strains result in a temperature

variation, but since the inhomogeneous medium consists
of two components, an overall temperature difference ΔT
will be set up between these two components. This temperature
difference results in heat flow Q and a heat flow rate
$\dot{Q}$. As heat flows from the hotter to the colder component,
there will be a net increase of entropy, that is related
to the energy extracted from the elastic wave. Hence,
for the entropy change from a crest to a through of the
wave, from a region at temperature $\bar{T} + \Delta\bar{T}$ to a region
at temperature $\bar{T}$, for $\bar{T}$ the ambient temperature, one has
the rate of entropy change as:

$$\frac{d}{dt}\left[\frac{Q}{\bar{T} + \Delta\bar{T}}\right] - \frac{d}{dt}\left(\frac{Q}{\bar{T}}\right)$$

$$= \frac{dQ}{dt}\left(\frac{\Delta\bar{T}}{\bar{T}^2}\right) = \frac{dS'}{dt}$$

$$(3.130)$$

Now for a composite system, with one ingredient having
a thermal conductivity K_1 and a scale size a_1, and another
component having thermal conductivity K_2 and scale size
a_2, then the heat flow will be governed by the more resistive
component, i.e.,

$$\text{if} \quad \left(\frac{K_1}{a_1}\right) > \left(\frac{K_2}{a_2}\right)$$

$$(3.131)$$

then region 2 will govern the flow rate and vice versa. Hence in the heat conduction process one has, for $\dot{Q}$ given on a per unit volume basis,

$$\left(\frac{dQ}{dt}\right) = \left(\frac{k}{a}\right) \Delta \overline{T} = \kappa_{eff} \, k \, \Delta \overline{T}$$

(3.132)

$$\left(\frac{k}{a}\right) = \min\left(\frac{K_1}{a}, \frac{K_2}{a}\right)$$

(3.133)

where $(\kappa \text{ eff}) \, k = \frac{k}{a})$ is used to compare the inhomogeneous medium with the theory for the homogeneous case, since in the present instance, the temperature gradient occurs over a particle size rather than a wavelength as it does in the homogeneous case. Thus this term acts as a scale factor for the inhomogeneous case as opposed to the homogeneous case.

Now $\Delta \overline{T} \propto e$ and in particular,

$$\left(\frac{\Delta \overline{T}}{e}\right)\bigg|_{S'} = \gamma \left(\frac{\lambda'}{C_v}\right)$$

(3.134)

for an adiabatic change, with λ' the bulk modulus, Cv
the specific heat at constant volume, and γ a constant[7],
e, the strain.

Hence, for

$$\Delta \overline{T} = \left(\frac{\lambda'}{C_v}\right)\gamma e$$

(3.135)

It is necessary to consider some cases that determine
the conduction process.

(Case I) - This is the instance when there is sufficient
time for all the heat to flow out of the particle in a
half cycle and for the temperature to equilibriate.

Then,

$$\frac{dQ}{dt} = \left(\frac{K}{a}\right)\left(\Delta\overline{T}\right)\left(\frac{a^2\overline{V_1}}{\overline{V_1} + \overline{V_2}}\right)$$

(3.136)

from (3.130),

$$\Delta S' = \Delta Q \left(\frac{1}{\overline{T}+\Delta\overline{T}} - \frac{1}{\overline{T}}\right);$$

$$\Delta S' = \Delta Q \left(\frac{\Delta\overline{T}}{\overline{T}^2}\right)$$

(3.137)

[7] $\gamma = -(\beta\overline{TV})$, see Appendix 2 for derivation.

and from above,

$$\Delta E = \overline{T} \Delta S' \left(\frac{\Delta \overline{T}}{\overline{T}} \right) \Delta Q = \frac{1}{\overline{T}} \left(\frac{K}{a} \right) a^2 (\Delta \overline{T})^2 \cdot$$

$$\sim \left(\frac{\overline{V_1} \, t}{\overline{V_1} + \overline{V_2}} \right) \tag{3.138}$$

In this case, one is considering the entire process of conduction as taking place over a fraction of a half cycle, and effectively is dealing with small granules. $\left(\overline{V_1} << \overline{V_2} \right)$ One has,

$$\Delta Q = C \Delta \overline{T} \left(\frac{\overline{V_1}}{\overline{V_1} + \overline{V_2}} \right) \tag{3.139}$$

and,

$$\Delta E = \overline{T} \Delta S' = \left(\frac{\Delta \overline{T}}{\overline{T}} \right) \Delta Q$$

$$= \left(\frac{C_1 \overline{V_1}}{\overline{V_1} + \overline{V_2}} \right) \left(\frac{\Delta \overline{T}^2}{\overline{T}} \right) \tag{3.140}$$

and, the temperature difference is, from (3.134),

$$\Delta \overline{T} \left(\frac{\gamma_2 \lambda_2'}{C_2} - \frac{\gamma_1 \lambda_1'}{C_1} \right) e = \Delta \left(\frac{\gamma \lambda'}{C} \right) e \tag{3.141}$$

while,

$$e^2{}_{rms} = E / \overline{\lambda}$$

so that,

$$\left(\Delta \overline{T}\right)^2 \approx \Delta \left(\frac{\gamma \lambda'}{c}\right)^2 \left(\frac{E}{\lambda_2}\right)$$

(3.142)

So, per unit volume, per half cycle, one has,

$$-\frac{\Delta E}{E} \approx \left(\frac{1}{\lambda_2}\right)\left(\Delta\frac{\gamma\lambda'}{c}\right)^2 \left(\frac{c_1}{\overline{T}}\right)\left(\frac{\overline{V_1}}{\overline{V_1}+\overline{V_2}}\right)$$

(3.143)

and for the half cycle $\left(\pi/\omega\right)$

$$\left(-\frac{1}{E}\frac{dE}{dt}\right) \approx \left(\frac{1}{\pi}\right)\left(\frac{1}{\lambda_2}\Delta\frac{\lambda'\gamma}{c}\right)^2\left(\frac{c_1\overline{V_1}}{\overline{T}(\overline{V_1}+\overline{V_2})}\right)\omega$$

(3.144)

which is the case with $V_1 \ll V_2$.

(Case II) - Here one must consider the case when the particles are large and K is small, and there is not enough time for the temperature to equilibriate over a half cycle. Hence, by the diffusion considerations from the heat conduction equation, one has, the time required as:

$$\tau = \left(\frac{r^2}{\alpha'}\right)$$

(3.145)

for r the radius (size) of the particle and α' the diffusion constant.

It is necessary to examine the instances:

If $\tau > (\frac{\pi}{\omega})$, the time for half a cycle, then the amount of heat that can flow out, ΔQ is reduced by the factor

$$\left(\frac{(\pi/\omega)}{\tau}\right) = \pi/\omega\tau$$

since,

$$(\Delta Q)\ (\pi/\omega\tau)\ \text{heat flows out}$$

$$(3.146)$$

here, since, $\Delta E = \overline{\Delta T}/\overline{T}\ \Delta Q$ then the $(\frac{1}{E}\frac{dE}{dt})$ term is reduced by this same factor. Hence, in this instance,

$$\left(-\frac{1}{E}\right)\left(\frac{dE}{dt}\right) = \left(\frac{1}{\pi}\right)\left(\frac{1}{\lambda_2}\Delta\left(\frac{\gamma\lambda'}{c}\right)^2\left(\frac{c_1\overline{V_1}}{\overline{T}(\overline{V_1}+\overline{V_2})}\right)\left(\frac{\pi}{\tau}\right)\right)$$

$$= \left(\frac{1}{\tau}\right)\ \left(\frac{1}{\lambda_2}\right)\left(\Delta\left(\frac{\lambda'\gamma}{c}\right)^2\right)\left(\frac{c_1\overline{V_1}}{\overline{T}(\overline{V_1}+\overline{V_2})}\right)$$

$$(3.147)$$

Now, if $V_2C_2\Delta T < \Delta Q(\frac{\pi}{\omega\tau})$, i.e., the heat capacity limits the heat conduction, then again the governing term is the lesser factor. Then, the expression above is further reduced by the factor $(\frac{C_2V_2}{C_1V_1})$, but the frequency dependence remains the same, and in this instance for large particles one has

$$\left(-\frac{1}{\mathcal{E}}\frac{d\mathcal{E}}{dt}\right) = \left(\frac{1}{\gamma\lambda_2}\right)\left(\triangle\left(\frac{\lambda'\gamma}{c}\right)^2\right)\left(\frac{C_2\overline{V_2}}{C_1\overline{V_1}}\right)\cdot$$

$$\cdot\left(\frac{C_1\overline{V_1}}{\mp(\overline{V_1}+\overline{V_2})}\right)$$

(3.148)

whereas for the case where there is sufficient time (for small particles) for all the heat to conduct out, but the conduction is limited by the heat capacity of the cooler material, one has,

$$\left(-\frac{1}{\mathcal{E}}\frac{d\mathcal{E}}{dt}\right) = \left(\frac{1}{\mathcal{T}}\right)\left(\frac{1}{\lambda_2}\triangle\left(\frac{\gamma\lambda'}{c}\right)^2\left(\frac{C_1\overline{V_1}\omega}{\mp(\overline{V_1}+\overline{V_2})}\right)\left(\frac{C_2\overline{V_2}}{C_1\overline{V_1}}\right)\right)$$

(3.149)

Hence, there are in essence four cases for the conditions in the loss process. In all cases, the thermodynamic attenuation mechanism is first linear in ω and then becomes independent of ω. For typical powders, $t \approx 1$ second and lies just in the midrange of the crossover of frequency dependences.

Finally by combining all the losses, noting again, from the general definition,

$$L = T + V + S + F + A + B$$

(3.150)

and that in the general formulation this is,

$$\left(-\frac{dE}{dt}\right)_L = -\left(\frac{dE}{dt}\right)_T - \left(\frac{dE}{dt}\right)_V - \left(\frac{dE}{dt}\right)_S$$

$$- \left(\frac{dE}{dt}\right)_F - \left(\frac{dE}{dt}\right)_A - \left(\frac{dE}{dt}\right)_B$$

(3.151)

for the total loss occurring in the granular medium, then,
the general equation of motion now becomes the following,
being cast in the form for the granular (powder) medium
with losses. This nonlinear partial differential equation
may be combined with the general solution with the specific
losses for the powder to yield the complete powder wavefield.

$$\nabla \left[(\bar{\lambda} + 2\bar{\mu}) \nabla \cdot \vec{U} \right] - \nabla \times (\bar{\mu} \nabla \times \vec{U})$$

$$+ 2 \left[(\nabla \bar{\mu} \cdot \nabla \vec{U}) - \nabla \bar{\mu} (\nabla \cdot \vec{U}) \right]$$

$$+ 2 \left[\nabla \bar{\mu} \times (\nabla \times \vec{U}) \right] = \rho \frac{\partial^2 \vec{U}}{\partial t^2}$$

$$\frac{i}{\omega \vec{U}} \left[\begin{array}{l} \left(\frac{4\mu^2 a^2}{m^2 \bar{V}^2 \, \text{surf} \, G \bar{V}} \right) (\Delta V)^2 (\omega 4) e^{-2(g_2 - g_2')(nr - n'r')} \\[12pt] + \left(\frac{2\mu^2 a^3}{(V \, G)} \frac{(\Delta V)^2 (\omega 4)}{\pi m^2 \bar{V}^2 \text{bulk}} \right) e^{-2(k_2 - g_2')(nr - n'r')} \\[12pt] + \left(\frac{c k}{\lambda_2} \right) \partial \left(\frac{\gamma \lambda}{c \pi} \right)^2 \left(\frac{C_1 V_1 C_2 V_2}{T(V_1 + V_2) C_1 V_1} \right) \\[12pt] + \frac{1}{\lambda \text{solid}} C_{\text{solid}} \, T (\gamma - 8)^2 c^2 k^2 \tau \\[12pt] + \left(\frac{\rho r^2}{\beta} \right) \left(\frac{\partial \lambda}{\lambda} \right) \left(\frac{c^2 k^2}{k \vec{U}} \right) \\[12pt] + \left(\frac{C}{V} \right) \mu \gamma (A_c r) \left[k^2 - \frac{2\mu P A_c}{1 + \frac{mcA}{k}} \right] \end{array} \right] \tag{3.152}$$

(d) Applications of the General Method to Field Behavior for Loss Mechanisms in the Powder

It is now important to treat the powder by the foregoing principles to deduce the wavefield description and consequently derive the nonlinear processes occurring in the propagation. For the functional form for the losses in the powder one has, from (3.152) the generalized loss function in the form:

$$\angle = Ak^2 + B(kx) + Ck$$

(3.153)

with A in E/ℓt units, B in E/vt units and C in E/At units $\bar{c}$ the velocity. Then, m = 2,3 in the formulation and one has,

$$U = e^{i\omega t} U_0 \left(1 + i(kx) + \left(\frac{(kx)^2 i^2}{2!}\right) \left[\frac{-e^{ikx}(Ak^2 + Ck)}{\omega U_0^2 (\bar{\lambda} + 2\bar{\mu}) k^2}\right] \right.$$
$$+ \frac{(kx)^3 i^3}{3!} \left[\frac{-e^{ikx}(Bk)}{U_0^2 k^2 (\bar{\lambda} + 2\bar{\mu})}\right]$$
$$\left. + \cdots \cdots \right)$$

(3.154)

which becomes,

$$U = U_0 \, e^{iwt} \left[1 + i(kx) + \frac{i^2(kx)^2}{2} \left[\frac{-e^{ikx}(Ak+C)}{(\lambda + 2\bar{\mu}) \, \mathcal{C} U_0^2 k^2} \right] \right.$$

$$+ \frac{(kx)^3 \, i^3}{3!} \left[\frac{-e^{ikx} \, (B)}{w U_0^2 (\lambda + 2\bar{\mu}) \, k^2} \right]$$

$$\left. + \quad \cdots \cdots \cdots \cdots \cdots \right]$$

(3.155)

and, in the present formulation,

$$U = U_0 \, e^{iwt} \, e^{ik_1 x \, D_N} \, e^{-k_2 x \, D_N}$$

(3.156)

for the D_N given as the general nonlinear term in the
expansion, (3.107).

This is the nonlinear behavior as a result of the
interaction processes occurring in the inhomogeneous granular
medium (powders surrounded by liquid, gas or vacuum) that
incorporates the effective Lamé constants as developed
prior (Eq. 3.16, 3.18, 3.20, 3.21) for the powder included
in the formulation. This example illustrates the departure
from the simplified exponential behavior of the weak,
linear attenuation case of the classical theory when nonlinear
losses are taken into account. In the particular case
here the powder has been dealt with by the series solution
to the general problem.

There remains one further loss to handle that is
present in the powder. This is the amplitude dependent
loss, V, (viscous loss) that is handled individually because
of its special nonlinear character and amplitude dependence.
Using the general form, Eq. (2.36) one has (with m=2),
since the power of x in the loss term, V is 0, so m-2
= 0 and m=2:

$$U = U_0\left(1 + ikx - (kx)^2\left[\frac{-e^{ikx}\,Ck}{2(\lambda + 2\mu)\,\omega\,U}\right] - i\frac{(kx)^3}{3!} + \ldots\right)$$

(3.157)

So that, bringing the third term in the series over to
the other side, the following appears:

$$U + \left(\frac{-e^{ikx}(Ck)\,U_0}{2(\lambda + 2\mu)\,\omega\,U}\right)(kx)^2 = U_0\left(e^{ikx} + (kx)^2\right)$$

$$= U_0\left(1 + i(kx) + \alpha(kx)^2 - \ldots\right)$$

(3.158)

One then has, making (3.158) a polynomial in U:

$$U^2 - U\left(U_0(kx)^2 + e^{ikx}\right) + \left[\frac{-e^{ikx}(Ck)\,U_0}{2(\lambda + 2\mu)\,\omega}\right](kx)^2 = 0$$

(3.159)

The solution for the amplitude from the quadratic is

$$U_{1,2} = \frac{U_0\left[((kx)^2 + e^{ikx}) \pm U_0\sqrt{\frac{(e^{2ikx} + 2(kx)^2 e^{ikx} + (kx)^4)}{-\left(\frac{(kx)^2(4e^{ikx} ck)}{\omega U_0(2(\bar{\lambda}+2\bar{\mu}))}\right)}}\right]}{2}$$

now,

$$\left(\frac{1}{E}\frac{dE}{dt}\right) = \left(\frac{ck}{E}\right) = \left(\frac{L}{E}\right)$$

$$\frac{dE}{dt} < E\omega$$

$$E << \left(\frac{\bar{\lambda}+2\bar{\mu}}{2}\right) \text{ since } e^2 \text{ small}$$

$$\left(\frac{dE}{dt}\right) << \tfrac{1}{2}\left(\bar{\lambda}+2\bar{\mu}\right)\omega$$

$$\Rightarrow \left[\frac{ck}{U_0\omega(2(\bar{\lambda}+2\bar{\mu}))}\right] < 1$$

So the approximation goes with the first term in the bracket and expanding the quadratic one has,

$$U = e^{i\omega t}U_0\left|(kx)^2 - (kx)^2\left(1 + \frac{e^{ikx}}{(kx)^2} + \frac{e^{2ikx}}{(kx)^4}\right)\right|$$

which becomes,

$$U = U_0 e^{i\omega t}\left(\frac{e^{ik_x}}{2} + \frac{e^{2ik_r}}{2(k_x)^2}\right)$$

(3.160)

Now,

$$k_2 = \alpha = \left(\frac{1}{c}\right)\left(\frac{1}{E}\frac{dE}{dt}\right) = \left[\frac{L}{c(\lambda + 2\mu)k_1^2 U_0^2}\right]$$

so, finally, substituting the above into (3.160):

$$U = U_0 e^{i\omega t}\left[e^{ik_1 x}\right]\left[e^{-\left[\frac{L}{c(\lambda + 2\mu)k_1^2 U_0^2}\right]x}\right]\left[1 + \frac{e^{ik_1 x}e^{-k_2 x}}{(k_x)^2}\right]$$

(3.161)

Hence, the total field may be obtained by superposition of these nonlinear solutions, (3.161) and (3.155).

The method can be applied to subsequent cases, some of which include the following:

(1) Lossy Fluids: Here the general wavefield description involves the substitution of $\bar{\mu}$ fluid/particles, and $\bar{\lambda}$ fluid/particles into the general series solution for the near and far fields, that incorporates losses. The resulting description describes the nonlinear propagation in the lossy fluids.

(2) Geophysical Scale Solutions: In this instance, for obstacles large in comparison with a wavelength, one employs the constitutive relationships with $\bar{\mu}$, $\bar{\lambda}$ for granules

and (liquid or gas) incorporated into the total solution
to obtain higher order scattering and diffraction effects
in the composite near field solution. Also additional
terms in the far field solution are used as further corrections
in order to account for the larger obstacles size.

(e) Lossy Mechanisms with a Distributed Forcing Function
     with Application to Piezoelectric Materials

     The general series method is not limited to acoustic
fields with no sources explicitly considered. The method
is employed now for cases with distributed sources, and
as a specific example one considers the case where a lossy
medium with a distributed forcing function is present.

     Here, $F = f(x)e^{+i\omega t}$, and, upon substituting this form
into the equation of motion, one has, for the lossy medium,
with distributed sources of the above form,

$$\left(\frac{\partial^2 U}{\partial x^2}\right)\left(\bar{\lambda} + 2\bar{\mu}\right) = \rho\left(\frac{\partial^2 U}{\partial t^2}\right) + L\left(\frac{\partial t}{\partial U}\right) - f(x)e^{+i\omega t}$$

(3.162)

This is so because the distributed forcing function contributes
to the motion of a point in the displacement field by
acting as a source of wave generation.

     By the same arguments as before, there arises, (I)
Re = Re

$$d(k) \, A_m'' (x^{m-2}) + f(k) \, B_q'' (x^{q-2})$$

$$+ \omega^2 \varrho \, j(k) (A_{m-2}) (x^{m-2}) + \omega^2 \varrho (B_{q-2}'') (x^{q-2}) S(k)$$

$$= \left[\left(- \frac{\sin kx}{\omega v_0^2} \right) L(x^{m-2}) + \left(\frac{f(x'') \cos \omega t}{v_0^2} \right) \right]$$

$$(3.163)$$

(II)   Im = Im

$$R(k) \, A_m'' (x^{m-2}) + P(k) \, B_q'' (x^{q-2})$$

$$+ \omega^2 \varrho \, (A_{m-2}'') (x^{m-2}) M(k)$$

$$+ \omega^2 \varrho \, E(k) (B_{q-2}'') (x^{q-2})$$

$$= - \left[\left(\frac{\cos kx}{\omega v_0^2} L(x^{m-2}) - \frac{f(x'') \sin \omega t}{v_0^2} \right) \right]$$

$$(3.164)$$

Now, $f(x) = \sum_{\text{all } m} A_m' x^m$ for any $f(x)$, since any arbitrary function can be represented as a power series. Since $q = m + 1$, one has a separated set of subseries. Then one arrives at the recursion relationship between the even-counted coefficients; this is done by the same arguments as before for the imaginaries:

$$R(k) \, A_m{}''(x^{m-2}) + w^2 \varrho \, A_{m-2}''(x^{m-2}) \, M(k)$$

$$= -\left[\left(\frac{\cos kx}{w v_o{}^2} L(x^{m-2}) - f \frac{(x) \sin w t}{v_o{}^2}\right)\right]$$

<div align="right">(3.165)</div>

And also, as before, for the odd-counted coefficients:

$$P(k) \, B_q{}''(x^{q-2}) + w^2 \varrho \, E(k) \, B_{q-2}'' (x^{q-2})$$

$$= -\left[\frac{\cos kx}{w v_o{}^2} L(x^{q-2}) - f \frac{(x^{q-2}) \sin w t}{v_o{}^2}\right)\right]$$

<div align="right">(3.166)</div>

and also, from the real set:

$$d(k) \, A_m{}'' (x^{m-2}) + w^2 \varrho \, j(k) \, A_{m-2}'' (x^{m-2})$$

$$= \left[\left(-\frac{\sin kx}{w v_o{}^2} L(x^{m-2}) + f \frac{(x^{m-2}) \cos w t}{v_o{}^2}\right)\right]$$

<div align="right">(3.167)</div>

$$f(k) \, B_q''(x^{q-2}) + \omega^2 \rho \, \overset{97}{B_{q-2}''}(x^{q-2}) \, S(k)$$

$$= \left(-\left(\frac{\sin kx}{\omega v_o^2} \angle(x^{q-2}) \right) + \left(f \frac{(x^{q-2}) \cos \omega t}{v_o^2} \right) \right)$$

(3.168)

Combining the reals and imaginaries, (3.168 and 3.166, and 3.167 and 3.165) one has the recursion relations between the even and odd coefficients:

$$d(k) \, A_m''(x^{m-2}) + \omega^2 \rho \, j(k) \, A_{m-2}''(x^{m-2})$$

$$= \left(-\frac{\angle(x^{m-2}) e^{ikx}}{\omega v_o^2} + \frac{e^{+i\omega t}}{v_o^2} f(x^{m-2}) \right)$$

(3.169)

and,

$$f(k) \, B_q''(x^{q-2}) + \omega^2 \rho \, B_{q-2}''(x^{q-2}) \, S(k)$$

$$= \left(-\frac{\angle(x^{q-2}) e^{ikx}}{\omega v_o^2} + \frac{e^{+i\omega t} f(x^{q-2})}{v_o^2} \right)$$

(3.170)

Again as before in the general case one is solving for the nonlinear dominant coefficients for the distributed forcing function. One has, the following for the A_m'' from the product terms in terms of the base coefficients, A_o, and Bq'' in terms of B_1 that are arbitrary: (letting them be 1)

$$A_m'' = \left[\frac{-\frac{e^{ikx}}{\omega \, v_0^2} \angle (x^{m-2}) \overset{98}{} + \frac{e^{+i\omega\tau} f(x^{m-2})}{v_0^2}}{(m)(m-1) \sum\limits_{N'=0}^{m} (k_1)^{m-N'} i^{N'} (k_2)^{N'} (\bar{\lambda} + 2\bar{u}) \binom{m}{N}} \right] \tag{3.171}$$

And also for the Bq" coefficients:

$$B_q'' = \left| \frac{-\frac{e^{ikx}}{\omega \, v_0^2} \angle (x^{q-2}) + \frac{e^{+i\omega t} f(x^{m-2})}{v_0^2}}{(q)(q-1) \sum\limits_{R''=1}^{q} (i^{R''+1})(k_1^{q-R''})(k_2)^{R''} \binom{q}{R}(\bar{\lambda} + 2\bar{u})} \right| \tag{3.172}$$

Finally, regrouping, one has the following expression for the general wavefield solution incorporating the effects of a distributed forcing function:

$$U = \left(U_0 \, e^{i\omega t} \right) \left(\sum_m A_m''(kx)^m + i \sum_q B_q''(kx)^q \right)$$

$$= U_0 \left(e^{i\omega t} \right) \left(1 + i \left(k_1 + i k_2 \right) x \right)$$

$$+ U_0 \, e^{i\omega t} \left[\sum_{\substack{m=2 \\ even}} \frac{(kx)^m \left(-\frac{e^{ikx}}{\omega v_0^2} \angle (x^{m-2}) + \frac{e^{+i\omega t} f(x^m)}{v_0^2} \right)}{(m)(m-1) \sum\limits_{N'=0} (k_1)^{m-N'} (k_2)^{N'} \binom{m}{N}(\bar{\lambda}+2\bar{u})} \right.
$$

$$+ e^{i\omega t} U_0(i) \left[\frac{\sum_{\mathcal{g}=3}^{\infty} \left((kx)^{\mathcal{g}} \left[\frac{-e^{ik_x L(x^{\mathcal{g}-2})}}{\omega v_0^3} + \frac{f(x^{\mathcal{g}-2})}{v_0^2 \cdots} \right] \right)}{(\mathcal{g})(\mathcal{g}-1) \sum_{k''=1}^{\prime\prime}\binom{\prime\prime}{\lambda}(\lambda+2\alpha)k_i^{\mathcal{g}}\frac{k''}{k_i k''}} \right]$$

(3.173)

Where the nonlinear terms represent higher order interaction and attenuation processes. The wavefield U_x described above represents the solution inside the lossy material with distributed forcing function.

Now one knows that,

$$\left(\frac{dU}{dx}\right) = P(x) E = P(x) E_0 e^{i\omega t}$$

(3.174)

for $P(x)$ the piezoelectric coefficient and E the electric field, and $\frac{dU}{dx}$ the strain.

Now,

$$F = \lambda' E_0 \left(\frac{dP}{dx}\right) e^{i\omega t} = \lambda' E \left(\frac{d^2 U}{dx^2}\right)$$

(3.175)

so that one associates F with $\frac{dP(x)}{dx}$, and F(x) is the forcing function produced by the electric field E.

Using the above considerations, one may display an example of the general techniques with regards to a piezo-electric material with P(x) defined as

$$P(x) = K \quad (\text{constant}) \quad a \leqq x$$

$$(3.176)$$

$$P(x) = Cx \qquad 0 \leqq x \leqq a$$

$$(3.177)$$

$$P(x) = 0 \qquad\qquad x < 0$$

$$(3.178)$$

Then, for F(x) the distributed forcing function one has,

$$F = E \lambda' \left(\frac{dP}{dx}\right) = C\lambda' E \qquad 0 \leqq x \leqq a$$

$$F = 0 \qquad \text{otherwise}$$

$$(3.179)$$

The general nonlinear solution in the piezoelectric material becomes, applying Eqs. (3.173) and (3.179),

$$U_{interior} = e^{i\omega t} U_0 \left(1 + i(kx) + \frac{(kx)^2 i^2}{2!} \right) \cdot$$

$$\cdot \left[\frac{-\left(e^{ikx}/\omega \, U_0^2\right) L(x^0) + B\lambda' C E \, e^{i\omega t}}{U_0^2 \, k^2 \, (2(\tilde{\lambda} + 2\bar{\mu}))} \right]$$

$$- \frac{i(kx)^3}{3!} + \ldots \sim U_0 \, e^{ik_1 x} A_\sim - k_2 x A_\sim e^{i\omega t}$$

$$(3.180)$$

for A and B units factors.

Now it is known, that a solution of the form

$$U_{out} = U_0 \, e^{ik_1 x} \, B_N \, e^{-k_\sim x} \, B_N \, e^{i\omega t}$$

exists outside in the near field of the piezoelectric material.

Hence, for the transmitted wave, one has, applying the boundary conditions at the interface, (x=0)

$$U_T = U_0 - \text{Unreflected} \qquad U_i \Big|_{x=0} = U_0 \Big|_{x=0}$$

(3.181)

and,

$$ik_{out} B_N U_T = ik A_N U_0 + ik A_N U_R$$

$$\frac{d U_{out}}{dx}\Big|_{x=0} = \frac{d U_{it}}{dx}\Big|_{x=0}$$

(3.182)

so that, substituting (3.181) into (3.183), one has,

$$ik B_N (U_0 - U_R) = ik A_N (U_0 + U_R)$$

$$k_{out} B_N = k_{in} A_N \left(\frac{U_0 + U_R}{U_0 - U_R} \right)$$

$$k_{out} B_N = k_{in} A_N \left(\frac{1+R}{1-R} \right)$$

(3.183)

$$\widetilde{T} + \widetilde{R} = 1 \quad ; \quad 1 + \widetilde{R} = 2 - \widetilde{T}$$

for the reflection and transmission coefficients, $\widehat{R}$ and $\widehat{T}$; so,

$$\left(k_{out} \, B_N \right) = \left(k_{in} \, A_N \right) \left(\frac{2 - \widetilde{T}}{\widetilde{T}} \right) = k_{in} \, A_N \left(\frac{2}{\widetilde{T}} - 1 \right)$$

$$(3.184)$$

Since one cannot have simultaneous conservation of wave-vector and frequency,[8] one leaves (k_{in}/k_{out}) as a term in U.

The output can be described as:

$$U_{out} = U_0 \, e^{i \left(\frac{k_{in}}{k_{out}} \right) A_N \left(\frac{2}{\widetilde{T}} - 1 \right) X} \, e^{i \omega t}$$

$$(3.185)$$

where $\dot{T}$ is measured at the output of the piezoelectric material with a known input Uo. This is the nonlinear output from the peizoelectric material into the surrounding medium in the near field, with the general nonlinear term A_N in the series expansion described as:

[8]J. de Klerk, P. Klemens, E. Kelly, "Multilayer Enhancement of Microwave Piezoelectric Conversion in CdS - SiO Layers" Applied Physics Letters, 9,10 (Nov. 15, 1965.)

$$A_N = \left[\frac{-\frac{e^{ikx}}{\omega v_0^2} L\left(x^{N-2}\right)}{(\bar{\lambda} + 2\bar{\mu})(k^N)(N)(N-1)} \right]$$

(3.186)

Hence, the method may be used in cases where distributed sources (and sinks, as in the loss function itself) are present.

(f) Energy Loss Mechanism Size Ordering

for the Powder

It is extremely valuable to know how the losses order themselves over the frequency regime. This is so since, if one were to desire to obtain a first order description to the physical powder situation, ordering of the losses would permit the formulation of an approximate, simplified equation of motion and constitutive relations and allow one to deduce the first order description to the wave field.

Hence, one orders the losses over the frequency range to see their dominance to the contribution to L. This will enable one to obtain such orders of approximation to any desired degree.

(Case I): In the short wave length limit, $(\lambda \leq R,$ k large) one has, to order N,

$$S \propto k^{4+N} \qquad F \propto k^2$$

$$V \propto k/v \qquad A \propto k^2$$

$$T \propto (Constant) \qquad B \propto k^2$$

$$(3.187)$$

To lowest order, one puts N = 0 into the expression and gets the magnitude dominance:

$$S > F > B > A > V > T$$

$$(3.188)$$

(Case II): In the long wavelength limit, ($\lambda > R$, k small) one has,

$$S \propto k^4 \qquad F \propto k^2$$

$$V \propto k/v \qquad A \propto k^2$$

$$T \propto k \qquad B \propto k^2$$

$$(3.189)$$

For weak scattering, little scattering loss occurs for low k. For N = 0 in lowest order $S \propto k^4$ and the familiar Rayleigh scattering dependence is regained.

So here one has the dominance as:

$$V > T > F > B > A > S$$

$$(3.190)$$

So that according to the frequency regime that is investigated, the losses reorder themselves in dominance for the two regimes, $\lambda < R$ and $\lambda \geq R$ for R the size of the obstacle. Hence in the two cases, by ordering of the largest to smallest, one discovers in the two regimes, the partitioning:

$$\angle_{\lambda > R} = (V + T + F + B + A + S)$$

(3.191)

and

$$\angle_{\lambda < R} = (S + F + B + A + V + T)$$

(3.192)

As a result of this ordering, one sees that in the frequency regimes of $\lambda > R$ and $\lambda \leq R$ one may approximate L by the first few terms,

$$\angle_{\lambda > R} \approx (V + T + F)$$

(3.193)

and,

$$\angle_{\lambda \leq R} \approx (S + F + B)$$

(3.194)

So, effectively, this means that in the two frequency regimes, the following apply,

$$\angle_{low\ frequency} \sim \left(\frac{k}{v} + k \right)$$

(3.195)

and,

$$L_{High\ frequency} \sim (k^4 + k^2)$$

(3.196)

So that the low frequency regime has $L \sim (k/u)$ and the high frequency regime has $L \sim k^4$. This shows that the largest attenuation is at the higher frequencies and the energy losses go up in powers of k^4. In the intermediate frequency regime, $\lambda \approx R$, all losses are important, and a direct ordering for truncation would not be advisable. In this case, all losses should be used in the general equations of motion to develop the wavefield.

CHAPTER IV

Power Series Solution Capabilities

as Opposed to Perturbation Analysis

and Crossover Criteria

(a)   Limitations on Perturbation Theory

The concept of a series solution is amenable to many
intractable problems since admissable physical functions
are in essence, convergent series of polynomials. Hence
it is not surprising, that, when the complexity of a problem
surpasses the small storehouse of analytical closed form
solutions, a series solution is developed to the equation
at hand. One can now examine the capabilities of the
series as a satisfactory means for the development of
a solution.

One investigates the limits of perturbation theory.
For weak scattering, perturbation theory is acceptable,
but for strong scattering, it fails. For $\lambda \leq R$ perturbation
theory fails since the wave is refracted by the obstacle.
Perturbation theory also fails for strong scattering when
the mean free path as calculated by the theory is comparable
to or less than a wavelength. For $\lambda > R$, in general,
perturbation theory is permissible for the weak scattering

process. In the present treatment, the loss mechanism
L has been formulated as:

$$L = S + \begin{bmatrix} \text{other nonlinear energy loss mechanisms} \\ \text{not handled adequately be perturbation} \\ \text{theory} \end{bmatrix}$$

(4.1)

S is the scattering loss mechanism and for $\lambda > R$
the perturbation theory determination of S is acceptable,
but for other cases, the theory is not adequate, and another
approach must be developed.

Now since perturbation theory fails in strong scattering
then one must use an approach whereby the displacement
field may be expanded into a series form to obtain the
effects for strong scattering and high losses. Hence,
where the perturbation theory was used to calculate weak
scattering and the nonlinear losses were handled by series,
strong scattering and general nonlinear losses would incorporate
a series throughout. In this light, one constructs Table
IV, as a result of these considerations:

TABLE IV

Regions of Applicability of Solutions	
Scattering Mechanisms	Method
Weak	Perturbation Theory (P)
Strong	Series Solution (S)

Other Generalized Losses[9]

Series Solution (S)

Lw = P + S = Perturbation Theory and Series

Ls = S = Series Solution

Now for an inhomogeneous lossy medium, at the high end of the frequency scale, scattering effects appear to be a dominant mode of removing energy from the wave. These scattering effects become more and more significant as one increases in frequency. At the lower end where scattering effects are not as important (wavelength greater than obstacle size) attenuation takes place moreso due to the "mechanical" processes (viscous and frictional and thermodynamic losses). Hence the dominant attenuations at the low to intermediate frequency regime are those "mechanical" loss processes with scattering playing a secondary role.

[9] These include the terms T, V, A, B, F defined earlier in the case of the powder. In the general case these include any losses of nonlinear character whereby the perturbation theory is not adequate.

Hence as the wavelength begins very large, loss processes
in the inhomogenous medium are small; as the wavelength
decreases, the loss processes begin to play an increasingly
more dominant role where "mechanical" losses dominate
at the intermediate end ($\lambda \approx R$) and then as one approaches
$\lambda < R$, scattering effects predominate. As a result one
may construct the table below:

TABLE V

Dominant Loss Processes Occurring Over Frequency Regime	
$\lambda \gg R$	(Major) (losses very small except for small atten- uation)
$\lambda \geq R$	(losses are mechanical in nature) V - (1) Viscous Loss T - (2) Thermodynamic Loss F - (3) Frictional Loss S - (4) Weak Scattering Loss (Perturbation Theory Applicable)
$\lambda < R$	Strong Scattering Losses Multiple Scattering - intrinsic attenuation in the solid/liquid (A,B)

Energy loss is very strong at wavelength $\lambda \leq R$ in the spectrum. Thus there must be a sort of "characteristic" frequency ω_c, below which the wave passes, above which it is strongly attenuated. This ω_c is directly related to the properties of the medium in terms of its mechanical variations and elastic constants. For a perfectly homogeneous medium on the other hand, the wave is only weakly attenuated by anharmonicities at all frequencies and no dispersion takes place, whereas for the inhomogeneous, lossy, medium, dispersion occurs and losses take place. The critical frequency ω_c depends upon the properties of the medium, ie, $\omega_c = \omega_c(L)$, and the characteristic frequency is implicitly loss dependent.

A class of problems of interest to the scientific community in physics, geophysics, and engineering has been examined whereby energy loss mechanisms are considered in the degradation of an elastic wave propagating in an inhomogeneous medium. For the most part idealized solutions give insight into general behavior, but sidestep many of the important physical processes taking place in actual propagation. The general theory is consistent with the special case of low-loss homogeneous media, with an oscillatory solution in the limit, but at the same time, gives an extension of the theory as it exists today to allow one to deal with energy loss mechanisms occurring in a granular, lossy medium. This same approach may be used to obtain solutions to any inhomogenous medium by formulating the

appropriate "loss function" L and developing a series
solution consistent with the resulting inhomogenous partial
differential equation; hence the method has broad application
in its scope to similarly related problems. This method
should provide results in the fields of acoustics, elasticity,
electromagnetics, geophysics and physics if the method
is correctly formulated and the problem admits a solution.

(b) Physical Situations Requiring
the General Series Formulation

At this point it is of value to indicate a few physical
situations that would require a generalized series solution
as opposed to a perturbation technique. Such cases include:
(1) Lossy fluids - In this case with viscous and amplitude
dependent losses, nonlinearities of a general nature occur
and the series is needed. The behavior is described by
the general series.
(2) Elastic wave propagation in mud/muddy rivers - In
this geophysical situation, multiple scattering and viscous
and frictional loss mechanisms require the general series.
No actual truncation would be used as a result of turbulence
effects.
(3) Strong scattering of ultraviolet light - The general
series is needed in this physical situation as perturbation
theory fails for strong scattering and $\lambda \leq R$. Weak scattering

would be handled by perturbation theory.

(4) <u>Wave propagation in icy oceans</u> - Here, $\lambda \leq R$ for large pieces of floating ice on the ocean and the thermo-dynamics loss is large. The general series would be used.

(5) <u>Wave propagation on sandy bottoms of lakes/oceans</u> - In this case surface effects of frictional loss are present and general series is needed. No <u>truncation</u> would be used.

(6) <u>Dampening of vibrating beams in concrete beds</u> -Here energy is highly damped and strong attenuation occurs. The theory can be used to determine attenuation distance in the bed for determination of bed dimensions.

(7) <u>Acoustical tile thickness - optimum design for minimum cost</u> - Theoretically, one may obtain a good estimate of best thickness of acoustical tiles for manufacturning to design for cost effectiveness. An optimum thickness for dampening and minimum cost could be obtained.

Hence as is seen here, the general series formulation accounts for a wide class of problems that perturbution theory cannot. In this way, the general series provide an extension of the perturbation theory be augmenting the class of problems that can be handled.

CHAPTER V

Conclusion

In this investigation a development of a generalized
method to solve nonlinear wave equations with energy loss
mechanisms in an inhomogenous, medium has been made.
Specific application of the method has been made for a
granular lossy medium (powder) with vacuum, air or liquid
as an environment.

The power series solution to the problem yields insight
into the total energy absorption processes occurring through
the concept of higher-order attenuation processes as a
result of multiple interactions. The idea of the attenuation
coefficient in wave propagation has been broadened through
the introduction of the attenuation order interaction
coefficients. The amplitude and frequency of the propagating
wave is described in the near and far fields in consonance
with functional behavior observed such as damped electric
and acoustic signals on oscilloscopes, in lossy media.
In addition, the effect of loss on dispersion indicates
frequency shifting due to the loss mechanisms in the medium.

In summary, one concludes that the first order approxi-
mation of an attenuated wave

$$U = U_0 \left(e^{i\omega t}\, e^{ik_1 x}\, e^{-k_2 x} \right)$$

(5.1)

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is not sufficient for a general description, and that
the concept of attenuation order interaction coefficients
must apply, ie., in general representation,

$$U = U_0\, e^{i\omega t}\, e^{-k_2 x\, D_N}\, e^{i k_1 x\, D_N}$$

(5.2)

in the near field and, in general,

$$U = U_0\, e^{i\omega t}\, e^{i k_1 x}\, e^{-k_2 x\, D_N}$$

(5.3)

in the far field, where proper account of the energy loss
mechanisms has been taken in the construction of the nonlinear
attenuation order interaction coefficients, D_N.

APPENDIX 1

Attenuation Order Interaction
Coefficients in Lossy Media

If one considers the wavelike function, omitting the time dependence,

$$e^{i(k_1 + ik_2)x} = 1 + i(k_1 + ik_2)x + \left(\frac{i^2}{2!}\right)(k_1 + ik_2)^2 x^2$$
$$+ \left(\frac{i^3}{3!}\right)(x^3)(k_1 + ik_2)^3 + \ldots \ldots$$

$$= \sum_N \left(\frac{i^N}{N!}\right)(k_1 + ik_2)^N x^N$$

$$= \sum_{\substack{M=2N \\ even}} x^M \left(\frac{i^M}{M!}\right)(k_1 + ik_2)^M$$
$$+ \sum_{\substack{P=2N+1 \\ odd}} x^P (k_1 + ik_2)^P \left(\frac{i^P}{P!}\right)$$

$$= \sum_{M=2N} A_m'(kx)^M + (i) \sum_{P=2N+1} B_P'(kx)^P$$

$$(A1-1)$$

So for this "linearly attenuated" case, $(e^{ik_1 x} e^{-k_2 x})$ the coefficients A_m' and B_P' do not depend upon frequency or time or loss mechanism L. In this case the coefficients A_m', B_P' are constants, and the attenuation coefficient $\alpha = k_2$ only.

Consider now, the case,

$$U = U_0 \left(\sum_m A_m (kx)^m + (i) \sum_q B_q (kx)^q \right)$$

$$(A1-2)$$

where one encounters nonlinear interactions occurring
in the loss mechanisms over the entire frequency range
ω. Now, if

$$A_m = A_m\left(L, k, t\right) \quad ; \quad B_q = B_q\left(L, k, t\right)$$

<div align="right">(A1-3)</div>

no longer constants, one sees that,

$$U = U_0 \left(\sum_m A_m (kx)^M + (i) \sum_q B_q (kx/8) \right) e^{i\omega t}$$

<div align="right">(A1-4)</div>

$$U = U_0 \, e^{ik_1 x \, D_N} \, e^{-k_2 x \, D_N} \, e^{i\omega t} \quad (representation)$$

$$U = U_0 \, e^{i\omega t} \left(\sum_N \frac{(i\,kx)^N}{N!} \, D_N \right) \quad (near\ field)$$

$$U = U_0 \, e^{i\omega t} \, e^{ik_1 x} \left(\sum_N (-i)\frac{(kx)^N}{N!} \, D_N \right) \quad (far\ field)$$

$$with, \quad D_N = A_m \qquad m\ even \quad (M = 2N)$$
$$D_N = B_q \qquad q\ odd \quad (q = 2N+1)$$

in the general series expansion of U.
So one sees that the true attenuation coefficient α is
made up of $\alpha = k_2 D_N$ with D_N identified as the Nth coefficient
in the series expression solution to the nonlinear inhomo-
genous loss problem. In the former case, $\alpha = k_2 D_0 =$
k_2 as before, when higher order losses were not considered
and higher order attenuations neglected that cannot be
neglected in the nonlinear loss problem with interactions
of various higher orders. One calls the D_N the (nonlinear)
attenuation order interaction coefficients.

APPENDIX 2

Relationship Between Thermal Strain, Temperature Change and Thermodynamics Parameters

One has,

$$(C_p - C_v) = \frac{\beta^2 \overline{T} \overline{V}}{K} = \beta^2 \overline{T} \overline{V} \lambda'$$

(A2-1)

for λ' the bulk modulus, K the compressibility, β the expansivity, $\overline{T}$ the temperature and $\overline{V}$ the volume, with Cp, and Cv the specific heats at constant pressure and volume.

Now, $\overline{\gamma}' = (C_p / C_v)$

$$(C_p / C_v - 1) = (\overline{\gamma}' - 1) = \left(\frac{\lambda' \beta^2 \overline{T} \overline{V}}{C_v} \right)$$

(A2-2)

Now from the $(\overline{T}dS)$ equation, one can write, adiabatically, $(\overline{T}dS = 0)$

$$\left(\frac{d\overline{T}}{e} \right)\Big|_s = \left(\frac{d\overline{T}}{d\overline{v}} \Big|_s \right) = v \left(\frac{d\overline{T}}{d\overline{v}} \right)\Big|_s = \left(\frac{1 - \overline{\gamma}'}{\beta} \right) = \left(\frac{-\lambda' \beta^2 \overline{T} \overline{v}}{C_v \beta} \right) = \Gamma \left(\frac{\lambda'}{C_v} \right)$$

(A2-3)

Where, here,

$$\Gamma = -(\beta \overline{T} \overline{V})$$

(A2-4)

is a characteristic volume.

Here, a strain e develops a temperature difference $(d\overline{T})$ and the strain gradient causes a temperature gradient that leads to conduction loss, but in the inhomogeneous medium, the strain difference causes a temperature difference between the different components and this leads to conduction loss.

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APPENDIX 3

Relative Magnitudes of Losses in Numerical Approximation

One has for the thermodynamic loss in powders,

(I)
$$T = \left/ \left(-\frac{1}{E}\frac{dE}{dt} \right) \right| = \left(\frac{1}{\pi} \right)\left(\frac{1}{10^6} \right)\left(140 - 100 \right)^2 \cdot$$

$$\cdot \left(\frac{4.2 \times 10^7 (10^{-2})}{300} \right) \omega = 0.6\, \omega = 3.6\, f$$

$$\tag{A3-1}$$

so that,

$$\left(T \propto k \right)$$

(II) For the viscous loss, one has,

$$\left(-\frac{1}{E}\frac{dE}{dt} \right)(\omega\, \tau) = \text{Averaged rate of viscous loss over many cycles}$$

$$\left(-\frac{1}{E}\frac{dE}{dt} \right)_{viscous} = (\omega^2)\left(\frac{\Delta\bar{\lambda}}{\bar{\lambda}} \right)\left(\frac{\ell r^2}{\eta v} \right)$$

$$\tag{A3-2}$$

so that,

$$\left(-\frac{dE}{E\, dt} \right)_{viscous} = v\, \omega \left(\frac{\Delta\bar{\lambda}}{\bar{\lambda}} \right)\left(\frac{\ell r^2}{\eta v} \right)$$

and,

$$\left(V \propto k \middle/ v \right)$$

$$\tag{A3-3}$$

for water, $\ell = 1$; $\eta = 10$ $r = .1$ (cgs units)

$$u = 1 \text{ cm} \implies \ell = 10^{-4} = (k\,v)$$

$$E = \overline{\lambda}\, k^2\, v^2 = 10^{-8}\,\overline{\lambda}$$

$$S = (v\,E) = \left(10^4\right)\left(10^{-8}\,\overline{\lambda}\right) = 10^7 \text{ erg/cm}^2 \text{ sec}$$

$$\left(\frac{-1}{E}\frac{dE}{dt}\right)_{viscous} = (v\,\omega)\left(\frac{\Delta\overline{\lambda}}{\overline{\lambda}}\right)\left(\frac{a\,r^2}{\eta\,v}\right) = V$$

$$V = 10 \text{ f/sec} = \frac{1}{\tau_{viscous}}$$

$$(A3-4)$$

so for U > 1A° the viscous loss V dominates over T. So
that viscous loss is proportional to amplitude and goes
as f.

(III) For A, the intrinsic attenuation in the solid,
one has for rate processes,

$$\left(\frac{-dE}{E\,dt}\right)_A = \left(\frac{1}{\overline{\lambda}}\right)\left(C_v\,T\right)\left(\gamma - \overline{\gamma}\right)^2 \omega^2\,\tau \qquad (A3-5)$$

$$\frac{1}{\tau} \approx \left(\frac{\omega_{Debye}}{30}\right) \qquad \omega_D \approx \left(10^{11}\theta\right) \sim 3\times10^{13}/\text{sec}$$

$$\tau^{-1} \approx 10^{-12} \text{sec}$$

$$\left(\gamma - \overline{\gamma}\right)^2 \approx 2$$

$$\overline{\lambda} \sim \left(3\times10^{11}\right) \text{erg/cm}^3 \qquad C \sim 10^7 \text{ erg/deg cm}^3$$

$$\overline{T} \sim 300°K$$

$$\left(\frac{-dE}{E\,dt}\right)_A = A = \left(\frac{10^7\times300}{3\times10^{11}}\right)\left(2\times10^{-12}\right)\left(2\pi\right)^2 f^2$$

$$A = \left(8\times10^{-13}\right) f^2 \qquad (A3-6)$$

so that ($A \propto k^2$) for the intrinsic attenuation in the solid.

(IV) Now, for B, the intrinsic attenuation in the liquid, one has, for water,

$$\left(\frac{-d\varepsilon}{\varepsilon \, d\tau}\right)_B = \frac{1}{\lambda_{liq}} \, C_V \, \ell_{iq} \, \overline{T} \, (\gamma - \overline{\gamma})^2 \, \omega^2 \, \tau$$

with
$$(\gamma - \overline{\gamma})^2 \approx 100 \quad ; \quad C_V \approx 4 \times 10^7 \, erg/deg/cm^3 \qquad \text{(A3-7)}$$
$$\lambda = 3 \times 10^{10} \, erg/cm^2 \quad ; \quad \tau \approx 10^{-12} \, sec^{-1}$$

$$\left(\frac{-1}{\varepsilon} \frac{d\varepsilon}{d\tau}\right)_B = \left(\frac{(4 \times 10^7)(300)}{3 \times 10^{10}}\right)(100)(2\pi)^2 f^2 (10^{-12})$$

$$B = (1.6 \times 10^{-9}) f^2$$

so,
$$\left(B \propto k^2\right)$$
(A3-8)

Hence, at the low frequencies, one has little Akiesar loss.

(V) Now, for S, the scattering loss, in terms of the magnitude,

$$S \sim \left(\overline{V}^2 \omega^4\right)$$
(A3-9)

from Eq. (3.128) and (3.129), where $\overline{V}$ is the volume. The attenuation length ℓ goes as $\ell^{-1} = (NS)$ so that

$$\frac{1}{\ell} \propto \left(\frac{1}{\overline{V}}\right) \overline{V}^2 \omega^4 \sim \left(\overline{V} \, \omega^4\right)$$

$$\overline{V} = 10^{-3} m^3 \text{ for particle}$$

$$\lambda \sim 10^4 \, cm \qquad \ell^{-1} = (10^{-3})(10^{-16}) = \left(\frac{1}{v \tau}\right)$$

$$S = \left(\frac{-d\varepsilon}{\varepsilon \, d\tau}\right) = (10^{-19})(10^4) = 10^{-15}$$
(A3-10)

(VI) Lastly, for F, the frictional loss due to relative sliding between particles, one has,

$$\left| \text{grad}\,(P A\,r) - f'_{viscous} \right| = f'_{net}$$

(A3-11)

for A the particle area, r the particle radius, P the dynamic pressure, and F viscous the viscous force.

For sliding $f'_{net} > \mu\, P_H\, A$

for the surface, $P = 0$ $\quad$ at $z = 0$

(A3-12)

For 1 mm depth of sand, on a per unit area (cm^2) basis, one has, $\rho = 3\,gm/cc$; $\bar{V} = .1\,cc$; $M = \rho \bar{V} = .3\,gm/cm^2 mm$

$$F = m\,a = (- 3 \times 10^3)\,dyne$$

$$(\mu P A) = (- 3 \times 3 \times 10^{-4}{}_{\mu A}) = (10^{-4}/(mks))$$

$$P = \sqrt{E\,\chi'} = \sqrt{10^{11} \times 2\overline{00}} = (\sqrt{2})(3\times10^6) \approx 4\,atm$$

$$S = (1\,watt/cm^2) = (\bar{v}\,E) = E\left(\frac{10^5}{2}\right)$$

$$k \sim 10^{-4}\,cm^{-1} \qquad (kr) \sim 10^{-5} \qquad r = .1\,cm$$

$$\text{grad}\,(P A r) = (P k r) = 4 \times 10^{-5}$$

$$f = 1\,cycle/sec ; \quad w \sim 6 ; \quad v = 300\,m/sec$$

$$(\mu P A) = 10^{-4} \qquad (P k r) = .4 \times 10^{-4}$$

So that only down to .4 mm does the frictional force f' occur in measurable amounts even discounting viscous retards.

So that F is predominantly a surface effect for waves of low frequency. (for 1 cycle waves, F loss occurs only down to ½ mm or so, at higher k's, F of course goes deeper, linearly in k.) Now from Eq. (3.45):

$$F = \left(\frac{w}{v}\right) \int_{t_0}^{t} dt \sum_{j} (ka)\mu Y A_c \sum_{N=1}^{N/2} n\,\cos w\sin(kr - wt)$$

(A3-13)

$$F = \left(\frac{w}{v}\right) \sum_j (ka) \mu \int_{t_0}^{t} Y A_c \sum_{N=1}^{123} (n\cos\theta) w \sin(kx-wt)\, dt$$

$$F = \left(\frac{w}{v}\right) \sum_j (ka) \mu Y \int_{t_0}^{t} A_c \sum_{N=1}^{N/2} n(\cos\theta)\left[\cos(kx-wt) - \cos(wt_0 + kx)\right]$$

$$t_0 = t - \delta$$

$$F = \left(\frac{c}{v}\right) k^2 (a A_c) \mu Y \sum_{N=1}^{N/2} (n\cos\theta)\left[\cos(kx-wt) - \cos(kx - w(t-\delta))\right]$$

$$F = \left(\frac{c}{v}\right) k^2 (A_c r) \frac{\mu Y}{2\sqrt{3}}\left[\cos(kx-wt) - \cos(kx-wt)\cos\delta - \sin(kx-wt)\sin\delta\right]$$

Now, in general, the point t_0 is such that $t_0 = t_0(V, k)$
$\delta = \delta(V, k)$

Averaging over a quarter cycle where the variation occurs to yield a contribution,

$$\overline{F} = \left(\frac{c}{v}\right) k^2 (A_c r)\left(\frac{\mu Y}{2\sqrt{3}}\right)\left[\overline{\cos(kx-wt) - \cos(kx-wt)\cos\delta - \sin(kx-wt)\sin\delta}\right]$$

or, finally,

$$\overline{F} = \left(\frac{c}{v}\right) k^2 (A_c r) \frac{\mu Y}{2\sqrt{3}}\left[\frac{1}{2} - \frac{\overline{\cos}\delta}{2} - \frac{\overline{\sin}\delta}{2}\right] \quad (A3\text{-}14)$$

$$\overline{F} = \left(\frac{c}{v}\right) k^2 (A_c r) \frac{\mu Y}{4\sqrt{3}}\left[1 - \overline{\cos}\delta - \overline{\sin}\delta\right] \quad (A3\text{-}15)$$

now, in all cases, for sliding to occur,

$$\mu P A \leq |k^2 \sin(kx - wt_0) + m w A \sin(kx - wt_0)|$$

$$(\mu P A) \leq |k^2 + m w A| \sin(kx - wt_0)$$

$$(-wt_0 + kx) = \sin^{-1}\left(\frac{\mu P A_c}{k^2 + m w A}\right) \quad (A3\text{-}16)$$

for low k,

$$(\sin wt_0) = \left(\frac{\mu P_H A_c}{k^2 + m w A}\right) = \sin[w(t-\delta)] \quad (A3\text{-}17)$$

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at low k, threshold small $\Rightarrow \delta$ small $\Rightarrow$

$$\overline{\left(\frac{\mu P A_c}{k^2 + \eta \omega A}\right)} \sim \overline{\left(\sin \omega t\right) \cos \delta}$$

$$\overline{\left(\cos \delta\right)} \sim \left(\frac{2 \mu P A_c}{k^2 + \eta \omega A}\right)$$

A(3-18)A

At high k, threshold large $\Rightarrow \delta$ large $\Rightarrow$

$$\overline{\left(\frac{\mu P A}{k^2 + \eta \omega A}\right)} = \overline{\left(\cos \omega t \sin \delta\right)}$$

and $\sin \delta$ dominates, in the expansion of $\sin (\omega t - \delta)$ so that

$$\sin \delta \sim \left(\frac{2 \mu P A_c}{k^2 + \eta \omega A}\right)$$

A(3-18)B

in any event, one has from A(3-18)A and A(3-18)B, upon substitution into (A3-15),

$$\overline{F} = \left(\frac{C}{V}\right) k^2 \left(A_c r\right) \frac{\mu Y}{4\sqrt{3}} \left(1 - \frac{2 \mu P A_c}{k^2 + \eta \omega A}\right)$$

(A3-19)

Since Ac, P_H are so small the amplitude dependence in F is very weak for the present considerations. (f = 1 cycle/sec) Down to 1 mm of sand sliding occurs with the parameters as defined:

$$C = 2 \times 10^4 \, cm/sec$$

$$A_c = \left(.4 \times 10^{-8}\right) \left(\frac{4\pi}{4} \times 10^{-1}\right)^2 \sim .4 \times 10^{-10}$$

$$Y = 2000 \, atm \qquad ; \qquad r = .025 \, cm$$

$f = 1$ cycle/sec   depth to which sliding occurs = 1 mm sand

$P = 3 \times 10^{-4}$ atm $\Rightarrow$

$$d = \left(\frac{3 \times 10^{-4}}{2 \times 10^3}\right) \left(.025\right) = (4 \times 10^{-8}/cm$$

so that, on average

$$\overline{F} = (2 \times 10^4)(k^2)(4 \times 10^{11})(.025)(600 \times 10^6)\left(\frac{1}{7}\right)$$

$$\overline{F} = (10^{-2})(k^2)\left(\frac{600}{7}\right)(2) = 2k^2$$

$$\overline{F} = \left(\frac{(2)(36)\, f^2}{c^2}\right) \sim \left(\frac{72\, f^2}{4 \times 10^8}\right) = 1.8 \times 10^{-7} f^2$$

which yields,

$$\overline{F} = \left| \frac{-d\epsilon}{\epsilon\, dt} \right|_F \sim 1.8 \times 10^{-7} f^2$$

$$(F \propto k^2) \qquad\qquad \text{(A3-20)}$$

and this gives the order of the frictional loss and its frequency dependence.

APPENDIX 4

Derivation of the Generalized Inhomogeneous Media Equation of Motion with Energy Loss Mechanisms

Considering an element of volume (dxdydz) in the inhomogeneous medium, one has, for the displacement U(xyz,t) of the element (dxdydz) at the point x,y,z and the time t, that the inertial reaction of an element to an acceleration $\frac{d^2 U}{dt^2} \rho \, dx dy \, dz$ for ρ the density of the medium (ρ = ρ(xyz)), the net force on the element (dxdydz) to

forces acting on all faces is the sum over all faces of all force gradients. One has, indeed, summing,

$$\left(\frac{\partial \overline{F}}{\partial x}\right)\left(dx \, dy \, dz\right) + \left(\frac{\partial \overline{F}}{\partial y}\right)\left(dx \, dy \, dz\right) + \left(\frac{\partial \overline{F}}{\partial z}\right)\left(dx \, dy \, dz\right)$$
$$= \nabla \cdot \overline{F} \left(dx \, dy \, dz\right)$$

Then, one has the vectorial equation, noting $\nabla \cdot \overline{F}$ is a vector for dyadic $\overline{F}$, where $\overline{F}$ is the stress dyadic, and $\overline{F} = \overline{F}_x \overline{i} + \overline{F}_y \overline{j} + \overline{F}_z \overline{k}$ is a tensor of rank two.

$$\rho \frac{\partial^2 U}{\partial t^2} = \nabla \cdot \overline{F} = \nabla \cdot \left[\left(\overline{\lambda} \frac{|\nabla V + V \nabla|}{2} \right) \right] + 2 \overline{\mu} \left(\nabla U + U \nabla \right) \right] \quad \text{(A4-1)}$$

$$= \nabla \cdot \left(\overline{\lambda} \nabla U \right) + \nabla \cdot \left(\overline{\mu} \, U \nabla \right) + \nabla \cdot \left(\overline{\mu} \, \nabla U \right)$$

Now,
$$= \lambda \, \text{div grad } U + (\text{grad } U) \cdot (\text{grad } \lambda)$$
$$+ \overline{\mu} \, \text{div} \left(U \nabla \right) + U \, \text{div} (\text{grad } \overline{\mu})$$
$$+ \overline{\mu} \, \text{div grad } U + (\text{grad } \overline{\mu}) \cdot (\text{grad } U) \quad \text{(A4-2)}$$

$$\text{div} \left(U \nabla \right) = \text{grad div } U$$
$$\overline{\mu} \, \text{div} \left(U \nabla \right) = \overline{\mu} \, \text{grad div } U$$
$$\left(\overline{\lambda} + 2 \overline{\mu} \right) \text{div grad } U + \overline{\mu} \left(\text{grad div } U \right) \quad \text{(A4-3)}$$

Now,
$$= \left(\overline{\lambda} + 2 \overline{\mu} \right) \text{grad div } U + \overline{\mu} \left(\text{grad div } U \right)$$
$$= \left(\overline{\lambda} + 2 \overline{\mu} \right) \text{grad div } U$$

Also,
$$\nabla \overline{\mu} \times \left(\nabla \times U \right) = \left(\left(\nabla \overline{\mu} \cdot U \right) \nabla - \left(\nabla \overline{\mu} \cdot \nabla \right) U \right) = \nabla \overline{\mu} \left(\nabla \cdot U \right) - \left(\nabla \overline{\mu} \cdot \nabla \right) U$$

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Then, $(\nabla \bar{u})(\nabla \cdot \upsilon) + (\nabla \bar{u} \cdot \nabla) \upsilon = (\nabla \bar{u} \times (\nabla \times \upsilon)) + 2(\nabla \bar{u} \cdot \nabla \upsilon)$

$(\nabla \bar{u} \cdot \nabla) \upsilon = (\text{grad } \bar{u}) \cdot (\text{grad } \upsilon) = \nabla \bar{u} \times (\nabla \times \upsilon) + 2(\nabla \bar{u} \cdot \nabla \upsilon) - (\nabla \bar{u})(\nabla \cdot \upsilon)$

$\hspace{10cm}$ (A4-4)

Now $\quad \nabla \times (\bar{u} \nabla \times \upsilon) = (\nabla \cdot \upsilon) \bar{u} \nabla - (\nabla \cdot (\bar{u} \nabla) \upsilon)$

$\hspace{4cm} = (\nabla \cdot \upsilon) \bar{u} \nabla - \upsilon(\bar{u} \nabla \cdot \nabla + \nabla \cdot \text{grad } \bar{u})$

$\hspace{4cm} = \nabla \cdot \upsilon (\bar{u} \nabla) - \nabla(\nabla \cdot \bar{u}) \upsilon$

$\hspace{4cm} = (\nabla \cdot \upsilon) \bar{u} \nabla - \nabla \upsilon \cdot \text{grad } \bar{u}$

$\hspace{4cm} = -\upsilon \nabla \cdot \text{grad } \bar{u}$

$\nabla(\nabla \cdot \bar{u}) \upsilon = \upsilon \nabla(\nabla \cdot \bar{u}) = \upsilon \text{ div grad } \bar{u}$ $\hspace{3cm}$ (A4-5)

$(\upsilon \text{ div grad } \bar{u}) = -\nabla \times (\bar{u} \nabla \times \upsilon) = -(\nabla \times (\bar{u} \nabla \times \upsilon))$

Adding $+ ((\text{grad } \bar{u}) \nabla \cdot \upsilon)$ and subtracting $-(\text{grad } \bar{u} \nabla \cdot \upsilon)$, get

$(\text{grad } \bar{u})(\nabla \cdot \upsilon) + \text{grad } \bar{\lambda} (\nabla \cdot \upsilon) - \nabla \times \bar{u} (\nabla \times \upsilon)$

$- (\nabla \bar{u})(\nabla \cdot \upsilon) + (\nabla \bar{u}) \times (\nabla \times \upsilon)$

$+ 2(\nabla \bar{u} \cdot \nabla \upsilon) - \nabla \bar{u} (\nabla \cdot \upsilon)$

$(\bar{\lambda} + 2 \bar{u})(\text{grad div } \upsilon) + \nabla \bar{u} \times (\nabla \times \upsilon)$ $\hspace{2cm}$ (A4-6)

$\hspace{3cm} = \rho \dfrac{\partial^2 \upsilon}{\partial t^2}$

So,

$\text{grad}((\bar{\lambda} + 2 \bar{u})(\text{div } \upsilon)) - \nabla \times (\bar{u} \nabla \times \upsilon)$

$+ 2(\text{grad } \bar{u} \cdot \text{grad } \upsilon) - 2(\nabla \bar{u})(\nabla \cdot \upsilon) + 2(\nabla \bar{u} \times \nabla \times \upsilon)$

$\hspace{4cm} = \rho \dfrac{\partial^2 \upsilon}{\partial t^2}$ $\hspace{2cm}$ (A4-7)

but now one needs to account for losses. This is the vector wave equation of elasticity for inhomogeneous media[10] and agrees with that of Hook. One adds a dissipative force field to act as retarding loss to the force balance equation above, and account for "effective" Lamé parameters by the terms $\bar{\lambda}$ and $\bar{\mu}$.

The effect of the dissipative force field is to account for losses as the wave propagates through the inhomogeneous medium. One decribes the loss mechanism by the generalized

[10] J. Hook, "Separation of the Vector Wave Equation of Elasticity of Certain Types of Inhomogeneous Isotopic Media," Journal of the Acoustic Society of America, Vol. 33, No. 3, (March 1961).

"loss function" L.   L is in units of energy/unit volume/unit time, so, to convert to a dissipative force field one has $L/\overline{U} = L(\frac{\partial t}{\partial U})$ for $\overline{U}$ the particle velocity.   Hence, the final form as the starting point for the study of wave propagation in an inhomogeneous medium with energy loss mechanisms is

$$\nabla\left[(\overline{\lambda} + 2\overline{\mu})\nabla\cdot U\right] - \nabla\times\left(\overline{\mu}\left(\nabla\times U\right)\right)$$

$$+ 2\left(\nabla\overline{\mu}\cdot\nabla\right)U - 2\left((\nabla\overline{\mu})(\nabla\cdot U)\right)$$

$$+ 4\left((\nabla\overline{\mu})\times\nabla\times U\right) = \rho\frac{\partial^2 U}{\partial t^2} + L\left(\frac{\partial t}{\partial U}\right)$$

(A4-8)

which is a complete vector equation.

APPENDIX 5

Discussion of Rate Process and
Intrinsic Attenuation Processes

In order to develop the intrinsic attenuation in the solid or fluid, one may employ the following argument due to Klemens.[11]

In the low frequency range the wave causes a strain which alters the frequency of a mode q. The functional change is given by $\gamma(q)\in$ for $\gamma(q)$ the Gruneisen parameter (related to the frequency decrease in a solid arising from dilation), and which is essentially an anharmonicity coefficient. Since the frequency is changed because of the dilatation, the equilibrium occupation of a mode q is also changed, and so if the actual occupation number remains fixed, it will depart from the equilibrium state.

By balancing the rate of change of the occupation number of thermal mode due to strain and an accompanying temperature change against the rate of change due to dissipative processes (three phonon process) one has, as N is a function of ω, T, the frequency and temperature respectively,

$$\left(\frac{dN}{dt}\right)_{sound} = \left(\frac{dN^0}{d\omega}\right)\left(\frac{d\omega}{dt}\right) + \left(\frac{dN^0}{d\bar{T}}\right)\left(\frac{d\bar{T}}{dt}\right)$$

$$= \left(-\frac{n}{\tau}\right) = \left(\frac{dN}{dt}\right)_{interaction} \qquad (A5\text{-}1)$$

which is true in the relaxation time approximation and where n is a small deviation from the equilibrium occupation number N^0. Now for $X = \hbar\omega/kT$, one gets, from the above relation, (A5-1),

$$\left(dN^0/d\omega\right) = \left(dN^0/dx\right)\left(\hbar/kT\right)$$

[11]P. G. Klemens, "Effect of Thermal and Phonon Processes on Ultrasonic Attenuation", Physical Acoustics Principles and Methods vol. III-B Lattice Dynamics, edited by W. Mason, New York, NY: Academic Press, 1965).

and, $\dfrac{d\omega}{dt} = \left(\gamma \dfrac{d\epsilon}{dt}\right)\omega_0$

$\omega \sim \omega_0 (1 + \gamma\epsilon)$

and, $\dfrac{d\bar{T}}{dt} = \left(\dfrac{d\bar{T}}{d\epsilon}\right)\left(\dfrac{d\epsilon}{dt}\right) = i\,\omega_0\,\epsilon\,\dfrac{d\bar{T}}{d\epsilon}$

$$\left(\dfrac{dx}{d\bar{T}}\right)\left(\dfrac{dN^0}{dx}\right) = \left(\dfrac{dN^0}{d\bar{T}}\right) = \left(\dfrac{dN^0}{dx}\right)\left(\dfrac{-\hbar\omega}{k\bar{T}^2}\right) \qquad \text{(A5-2)}$$

$$= -\left(\dfrac{dN^0}{dx}\right)\left(\dfrac{x}{\bar{T}}\right)$$

Now one has, for a change of temperature with strain, that as a result of a changed strain $\delta\epsilon$, a departure of the phonon gas energy changes, and this change is equal to the energy change due to the change in temperature $\delta\bar{T}$ that results from $\delta\epsilon$, so that from (A5-2),

$$\sum_{\gamma} x\,\dfrac{dN^0}{dx}\,\gamma\,d\epsilon = \sum_{\gamma} x\,\dfrac{dN^0}{dx}\,\dfrac{d\bar{T}}{\bar{T}}$$

or, $\displaystyle\sum x\,\dfrac{dN^0}{dx}\left(\gamma\,d\epsilon - \dfrac{\delta\bar{T}}{\bar{T}}\right) = 0$

or, $\displaystyle\sum_{\gamma} x\left(\dfrac{dN^0}{dx}\right)\left(\dfrac{d\bar{T}}{d\epsilon}\left(\dfrac{1}{\bar{T}}\right) - \gamma\right) = 0$ $\qquad$ (A5-3)

Now in order to obtain absorption one must consider spatial variations of ϵ and these are thermal conduction processes. If all the γ's are not the same, and if they differ for different polarization branches the bracket alone cannot vanish separately for each mode. One must define an "average" γ, called $\bar{\gamma}$ for this as:

$$\left(\frac{1}{\bar{T}} \frac{d\bar{T}}{d\epsilon}\right) \sum_q x \frac{dN^0}{dx} = \bar{\gamma} \left(\sum_q x \frac{dN^0}{dx}\right)$$

$$= \sum_q \gamma x \frac{dN^0}{dx} \qquad (A5-4)$$

so that

$$\left(\frac{1}{\bar{T}} \frac{d\bar{T}}{d\epsilon}\right) = \bar{\gamma}$$

So, returning to equation (A5-1) with (A5-4) and (A5-2) substituted in, one has,

$$-\left(\frac{h}{T}\right) = \frac{dN^0}{dx}\left(\frac{\hbar\omega}{kT}\right)\gamma \dot{\epsilon} + \frac{dN^0}{dx}\left(\frac{-x}{T}\right)\frac{d\bar{T}}{d\epsilon}\frac{d\epsilon}{dt}$$

$$\left(-\frac{h}{T}\right) = x \frac{dN^0}{dx}\gamma (i\omega \epsilon) + \left(\frac{dN^0}{dx}\right)(-\gamma i\omega\epsilon) \qquad (A5-5)$$

$$h = i\omega_0 \epsilon (\gamma - \bar{\gamma}) x \frac{dN^0}{dx} \tau \qquad (A5-6)$$

with ω_0 the frequency of the sound wave.
Now the energy absorption is given by following, occupation number N change leading to energy change,

$$\sum_q (\hbar\omega) \frac{dN}{dt} = \sum_q \left(\sum_{q'} \sum_{q''} (\hbar\omega) R \left[(N+1)(N')(N'') - (N)(N'+1)(N''+1)\right]\right)$$

$$(A5-7)$$

The bracket vanishes when all N's as given by equilibrium value, ie,

$$\left[N_0' N_0 N_0'' - N_0 N_0' N_0''\right] = 0$$

$$(A5-8)$$

For deviations from equilibrium, one has terms linear and bilinear in n, the deviation, where $N = N^0 + n$. The linear terms do not lead to absorption because n, and ϵ vary periodically with time and vanish when averaged over one cycle of the wave. The bilinear terms become, in the deviation,

$$\left[(n+1)(n')(n'') - n(n'+1)(n''+1) \right]$$

$$= \left[nn'n'' + n'n'' - (nn'+n)(n''+1) \right]$$

$$= nn'\,n'' + n'n'' - nn'n'' - nn'' - n - nn' \qquad (A5\text{-}9)$$

$$= n'n'' - nn'' - nn' = n'n'' - n(n'+n'') \qquad (A5\text{-}10)$$

The last n is considered being averaged over a cycle of the wave.

Hence, the energy absorption rate becomes from (A5-7) and (A5-10),

$$\left(\frac{dE}{dt}\right) = \sum_g (\hbar w)\frac{dN}{dt} = \sum_g \sum_{g'} \sum_{g''} (\hbar w) R \left[(N+1)(N')(N'') - N(N'+1)(N''+1) \right]$$

and in bilinear form in the deviations, one has,

$$\left(\frac{dE}{dt}\right) = \sum_g \sum_{g'} \sum_{g''} (\hbar w) R \left((n'n') - n(n''+n') \right) \qquad (A5\text{-}11)$$

Now noting from (A5-6) one has,

$$n = i\,w_0\, \epsilon\, (r-\bar{F})\times \frac{dN^0}{dx}\, \tau \qquad (A5\text{-}12)$$

so that the bilinear terms that lead to absorption result in:

$$\left(\frac{dE}{dt}\right) = -\langle r-\bar{F}\rangle^2 w_0^2\, \tau^2 \left(\frac{dN^0}{dx}x\right)\left(\frac{dN^0}{dx}x\right)\epsilon^2 \qquad (A5\text{-}13)$$

Now, $$\left(\frac{dN}{dx}\right) \approx \left(\frac{N}{x}\right) \approx \frac{1}{x\,\tau}$$

because $$\left(\frac{1}{\tau} \sim N\right)$$

so that

$$\left(\frac{dE}{dt}\right) = -\langle r - \bar{r}\rangle^2 x\, \omega_0^2\, \epsilon^2\, \tau^2 \left(\frac{1}{x\tau}\right)\left(\frac{dN^0}{dx}x\right)$$

$$= -\epsilon^2 \langle r - \bar{r}\rangle^2\, \omega_0^2\, \tau \left(\frac{\hbar\omega}{kT}\right)\left(\frac{d}{d\tau}\,(N\hbar\omega)\right)\left(\frac{kT}{\hbar\omega}\right)^2$$

$$= -\langle r - \bar{r}\rangle^2\, \omega_0^2\, \tau\, \epsilon^2\, (C_v \bar{T})$$

$$= -\langle r - \bar{r}\rangle^2\, \frac{\omega_0^2\, \tau}{\lambda}\, (E)\,(C_v \bar{T})$$

since

$$E = \lambda\, \epsilon^2$$

so,

$$\left(-\frac{dE}{E\,dt}\right) = A = \langle r - \bar{r}\rangle^2_{solid}\left(\frac{\omega_0^2\, \tau}{\lambda}\right)(C_v \bar{T})$$

and,

$$\left(-\frac{dE}{E\,dt}\right) = B = \langle r - \bar{r}\rangle^2_{liquid}\left(\frac{\omega_0^2\, \tau}{\lambda}\right)(C_v \bar{T})$$

$$(A5\text{-}14)$$

This is the formula (A5-14) one uses for the intrinsic attenuation loss component in the composite. This may be applied separately to the liquid or the solid.

APPENDIX 6

Fog Attenuation Calcuation

An example of the thermodynamic loss As/Applied to fog $(V_1 \ll V_2)$ illustrates some of the principles discussed in this work.

Hence, for fog, from typcial values,

$$\left(\overline{V_1} \Big/ \overline{V_1} + \overline{V_2} \right) \approx 10^{-5} \tag{A6-1}$$

and,

$$\Delta\left(\frac{\gamma \lambda'}{c} \right) \approx \left(\gamma_2 \lambda_2' / c_2 \right) \qquad (\text{for gas phase})$$

so that,

$$\left(\frac{\gamma_2 \lambda_2'}{c_2} \right) = \left(\frac{1.4 \times 10^6}{(10^{-3})(10^7)} \right) = 140\, \%/\text{unit strain} \tag{A6-2}$$

$$\left(\frac{\gamma_2 \lambda_2'}{c_2} \right) \quad \text{dominate} \tag{A6-3}$$

as a result, one has, for T the thermodynamic loss,

$$T = \left(\frac{-d\epsilon}{\epsilon dt} \right)\Big|_T = \left(\frac{1}{\pi} \right)\left(\frac{1}{10^6} \right)(140)\left(\frac{(4.2 \times 10^7)(10^{-5} \omega)}{3\,00} \right) = .01\, \omega/\text{sec} \tag{A6-4}$$

for sound in fog, let $\omega = 600$ radians ($f = 100$ Hz), which is a typical low frequency source. Then,

$$\left(\frac{-d\epsilon}{\epsilon dt} \right)\Big|_T = 6\,\text{sec}^{-1} = \left(\frac{1}{\tau} \right) \tag{A6-5}$$

hence, the mean free path in fog becomes $\ell = 55\, \text{meters (dimæ)}$

for the sound at 100 Hz, leading to a reduction of (1/e) in level. And, for fog particles in air, of the order of 10 microns, one has,

$$t = \left(r^2 / \alpha^1 \right) = \left(\frac{10^{-6}}{10^{-3}} \right) = 10^{-3} \sec$$

(A6-6)

hence the crossover point for the thermodynamic attenuation mechanisms is f = 1000 cycles for fog and sound. This is for the frequency dependence of the thermodynamic loss, T.

One also notes from the buoyant force considerations in the fog,

$$F_{up} = \left(\overline{V} \varrho \, g \right) = \left(6 \pi \eta \, r \, \overline{v}^1 \right)$$

(A6-7)

Which describes the viscous Stokes force that is buoying up the fog particles.

Hence, for $\overline{v}$ = .1 cm/sec, the terminal velocity of fog particles one has,

$$\left(\frac{6 \pi \eta \, \overline{v}}{\varrho \, g} \right) = \left(\frac{V}{r} \right) = \left(\frac{4}{3} \pi \, r^2 \right) = 4 \times 10^{-11} m^2$$

$$r = 3 \; microns$$

Therefore the fog particles have a general lower limit in size of the order of 3 microns.

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